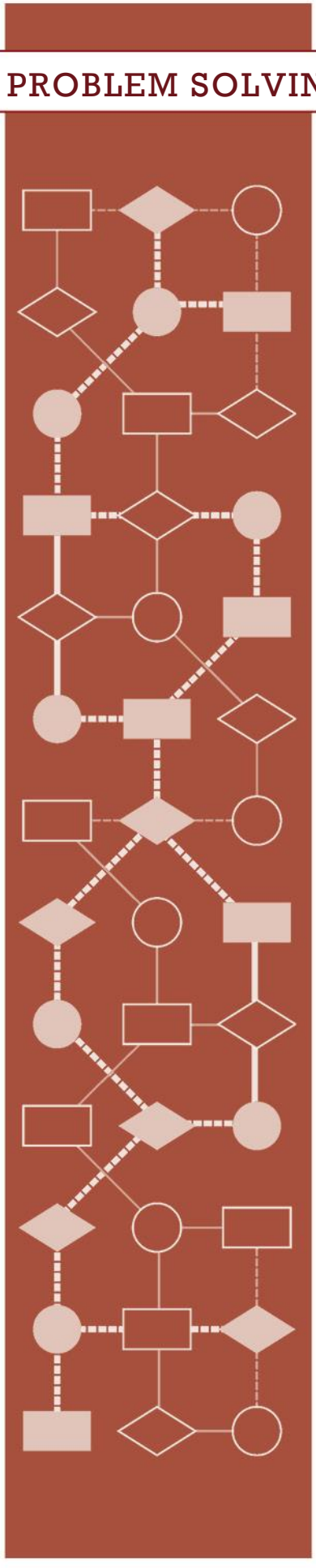
PROBLEM SOLVING



Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Maximizing Profit: *Selling Boomerangs*

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Maximizing Profit: *Selling Boomerangs*

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Interpret a situation and represent the constraints and variables mathematically.
- Select appropriate mathematical methods to use.
- Explore the effects of systematically varying the constraints.
- Interpret and evaluate generated data and identify the optimum case, checking it for confirmation.
- Communicate their reasoning clearly.

COMMON CORE STATE STANDARDS

This lesson relates to **all** the *Standards for Mathematical Practices* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 1, 2, 3, and 4:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

A-CED: Create equations that describe numbers or relationships.

A-REI: Solve equations and inequalities in one variable.
Represent and solve equations and inequalities graphically.

INTRODUCTION

This lesson is designed to help students develop strategies for solving optimization problems. Such problems typically involve using limited resources to greatest effect, as in, for example, the allocation of time and materials to maximize profit.

- Before the lesson, students attempt the problem individually. You then review their work and formulate questions for students to answer in order to improve their solutions.
- At the start of the lesson, students work alone answering your questions.
- Students are then grouped and engage in a collaborative discussion of the same task. In the same small groups, students are given sample solutions to comment on and evaluate.
- In a whole-class discussion, students explain and compare solution strategies seen and used.
- Finally, students revise their individual solutions and comment on what they have learned.

MATERIALS REQUIRED

- Each individual student will need a copy of the *Boomerangs* task, some plain paper, a calculator, and a copy of the *How Did You Work?* questionnaire.
- Each small group of students will need copies of the *Sample Responses to Discuss*.
- Graph paper should be kept in reserve and used only when necessary or requested.

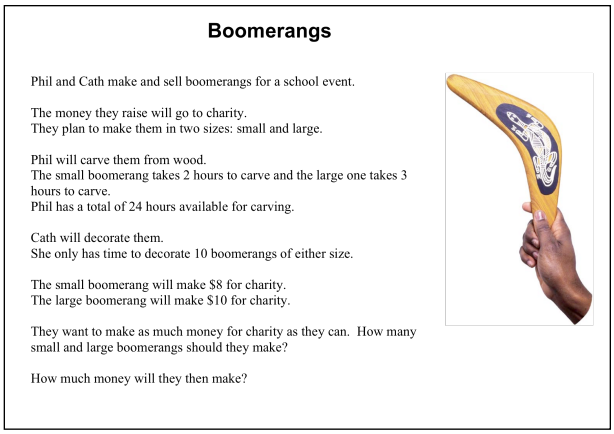
TIME NEEDED

Approximately 15 minutes before the lesson, a 1-hour lesson, and 10 minutes in a follow-up lesson.

BEFORE THE LESSON

Assessment task: *Boomerangs* (15 minutes)

Have the students do this task in class or for homework, a day or more before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.



Boomerangs

Phil and Cath make and sell boomerangs for a school event.

The money they raise will go to charity.
They plan to make them in two sizes: small and large.

Phil will carve them from wood.
The small boomerang takes 2 hours to carve and the large one takes 3 hours to carve.
Phil has a total of 24 hours available for carving.

Cath will decorate them.
She only has time to decorate 10 boomerangs of either size.

The small boomerang will make $8 for charity.
The large boomerang will make $10 for charity.

They want to make as much money for charity as they can. How many small and large boomerangs should they make?

How much money will they then make?

Give out the *Boomerangs* task and some plain paper. Introduce the task briefly to help the class to understand the problem and its context. You could show examples of boomerangs.

Boomerangs come from Australia where they are used as weapons or for sport.

When thrown, they travel in a roughly elliptical path and return to the thrower.

Boomerangs are made in many different sizes.

Now explain what students are to do.

Read through the questions and try to answer them as carefully as you can. Show all your work so that I can understand your reasoning.

As well as trying to solve the problem, I want you to see if you can present your work in an organized and clear manner.

It is important that, as far as possible, students are allowed to answer the questions without assistance.

Students who sit together often produce similar answers and then when they come to compare their work, they have little to discuss. For this reason, we suggest that when students do the task individually, you ask them to move to different seats. Then at the beginning of the formative assessment lesson, allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches. The purpose of doing this is to forewarn you of issues that will arise during the lesson itself, so that you may prepare carefully.

We strongly suggest that you do not score students' work. Research suggests that this will be counterproductive, as it will encourage students to compare scores and distract their attention from what they can do to improve their mathematics. Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on page T-4. These have been drawn from common difficulties observed in trials of this lesson unit. We suggest that you write a list of your own questions, based on your students' work. We recommend that you:

- write one or two questions on each student's work, or

- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these questions on the board when you return the work to the students at the beginning of the lesson. If students have used graphs or simultaneous equations in their solutions, add the relevant questions to their work. You may also want to note students with a particular issue, so that you can ask them about their difficulties during the lesson.

Common issues:	Suggested questions and prompts:
Has difficulty getting started	• What do you know? • What do you need to find out?
Makes an incorrect interpretation of the constraints and variables For example: The student has applied just one constraint, such as 'Phil has only 24 hours to make the boomerangs' or 'Cath can only make 10 boomerangs.' Or: The student has calculated the profit for making just one type of boomerang.	• What figures in the task are fixed? • What can you vary? • What is the greatest number of small/large boomerangs they can make? • Have you used any unnecessary restrictions on the number of small and large boomerangs to be made? • Why can't they make 50 boomerangs?
Works unsystematically For example: The student shows three or four seemingly unconnected combinations, such as 5 small and 5 large boomerangs, then 10 large.	• Can you organize the numbers of large and small boomerangs made in a systematic way? • What would be sensible values to try? Why? • How can you check that you remember all the constraints? • Do you cover all possible combinations? If not, why not? • How do you know for sure your answer is the best option? • Can you organize your work in a table?
Presents work poorly For example: The student presents the work as a series of unexplained numbers and/or calculations, or as a table without headings. Or: The student circles numbers and it is left to the reader to work out why this is the answer as opposed to any other combination.	• Would someone unfamiliar with your type of solution easily understand your work? • Have you explained how you arrived at your answer?
Has technical difficulties when using graphs/simultaneous equations For example: Lines are plotted inaccurately, axes not labeled or purpose of graph not explained. Or: A mistake is made when solving two correct simultaneous equations, or the correct solutions are obtained but no profit is calculated.	• Would someone unfamiliar with your type of solution easily understand your work? • How can you check your answer? • How do your answers help you solve the problem?
Produces a correct solution The student needs an extension task.	• Can you now use a different method? For example, a table or graph, or algebra? • Is this method better than your original one? Why? • In the problem investigated, how many boomerangs can be made in a month rather than 24 hours; would any method(s) be preferable to others?

SUGGESTED LESSON OUTLINE

Improve individual solutions to the assessment task (10 minutes)

Return to the students their assessment task papers and hand out calculators.

If you have not added questions to individual pieces of work, write your list of questions on the board (excluding the ones for graphs and simultaneous equations). Students are to select questions appropriate to their own work and spend a few minutes answering them.

Recall what we were looking at in a previous lesson. What was the task?

I read your solutions and have some questions about your work.

I would like you to work on your own for about ten minutes to answer my questions.

Students can either make a note of their answers to your questions on the back of their scripts or on a fresh piece of paper.

Collaborative small-group work (15 minutes)

Organize the class into small groups of two or three students and give out a fresh piece of plain paper to each group. Ask students to try the task again, this time combining their ideas.

Put your own work to one side. I now want you to work in groups. Your task is to produce a joint solution that is better than your individual solutions.

While students work in small groups you have two tasks: to note different student approaches to the task and to support student problem solving.

Note different student approaches to the task

Listen and watch students carefully. In particular, note any common mistakes. For example, are students consistently using all the constraints, or are they imposing unnecessary constraints? Also note whether students are using algebra and, if so, how they are using it. You can then use this information to focus a whole-class discussion towards the end of the lesson.

Support student problem solving

Try not to make suggestions that move students towards a particular approach to this task. Instead, ask questions that help students to clarify their thinking. You may discover that some students experience some difficulty in keeping more than one constraint in mind at a time. In that case, you may ask them to consider these questions:

If they were to make only small boomerangs, how much money would they make?

If they were to make two small boomerangs, how many large ones could they also make? How much money would they make?

For the first question, Cath's time is the limiting constraint, whereas in the second question, Phil's time is more significant. Students who organize their work into a table may choose to use column headings for 'Time needed for Phil' and 'Time needed for Cath' which they can use to check that both constraints have been met.

To help students really struggling with the task, use the questions on the previous page to support your own questioning. In particular, if students find it difficult to get started, you might like to ask:

What happens if they make three small and one large boomerang?

Can you try some examples? What would be sensible values to try? Why?

Can you organize the numbers of large and small boomerangs made in a systematic way?

If the whole-class is struggling on the same issue, write relevant questions on the board. You could also ask students who performed well on the assessment to help struggling students. If students are having difficulty making any progress at all, you could hand out two pieces of sample work to model problem solving methods.

Collaborative analysis of *Sample Responses to Discuss* (20 minutes)

After students have had sufficient time to attempt the problem, give each group of students a copy of each of the four *Sample Responses to Discuss* and ask for written comments. This gives students the opportunity to evaluate a variety of possible approaches to the task, without providing a complete solution strategy.

Imagine you are the teacher and have to assess this work. Make corrections and complete the work. Write comments on the accuracy and organization of each response.

Each of the sample responses poses specific questions for students to answer. In addition to these, you could ask students to evaluate and compare responses. To help them do more than check to see if the answer is correct, you may wish to use the projector resource *Evaluating Sample*:

What do you like about the work?

How has each student organized the work?

What mistakes have been made?

What isn't clear?

What questions would you like to ask this student?

In what ways might the work be improved?

You may decide there is not enough time for each group to work through all four pieces of sample work. In that case, be selective about what you hand out. For example, groups that have successfully completed the task using one method will benefit from looking at a different approach. Other groups that have struggled with a particular approach may benefit from seeing a student version of the same strategy.

During the small-group work, support the students as before. Note similarities and differences between the sample approaches and those approaches students took in the small-group work. Also check to see which methods students have difficulties in understanding. This information can help you focus the next activity, a whole-class discussion.

Whole-class discussion: comparing different approaches (15 minutes)

Organize a whole-class discussion to consider the different approaches used in the sample work. Ask students to evaluate the different solution methods one by one, then ask them to make comparisons.

Which approach is the most powerful? Why?

Which approach did you find most difficult to understand?

Encourage students to consider the relative merits of the different approaches (e.g. table, graph, algebraic approach), justifying their choice for the approach they liked best or found most difficult to understand.

In what way do you think this solution is better/worse than this one? Why?

See if students can identify similarities and difference in the sample responses:

What mathematics do the four sample responses have in common?

How is this mathematics used in different ways?

To critique the different strategies use the questions on Slide P-2: *Evaluating Sample Responses* and the worksheets: *Sample Responses to Discuss*.

Alex has realized that you have to take account of both constraints: Phil's time for making the boomerangs and Cath's time for decorating them. Alex has not examined different combinations of cases.

Phil can only make 12 small or 8 large boomerangs in 24 hours
12 small makes $96
8 large makes $80
Cath only has time to make 10, so $96 is impossible.
She could make 10 small boomerangs which will make $80.
So she either makes 8 large or 10 small boomerangs
and makes $80

Danny has found an effective way to organize his work, using a table. He has made some mistakes however. Part of the problem is that he loses track of the two constraints. It might have been helpful for him to include two additional columns headed: 'Time needed (≤ 24 hours)' and 'Total number made (≤ 10).' Then he could test each case and put a check mark if it satisfies both constraints.

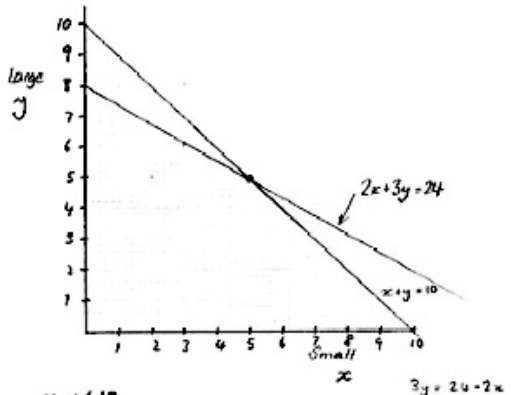
No of Small S	8S	No of large L	10L	Profit
0	0	8	80	80
1	8	~~9~~ 7	70	78
2	16	6	60	76
3	24	5	50	74
4	32	5	50	82 ←
5	40	4	40	80
6	48	3	30	78

The most Profit is $82

Jeremiah has tried an algebraic approach and has hit upon the correct solution. However, he has used equalities rather than inequalities. He needs to calculate the total profit to complete the question.

Small boomerangs = x
Large boomerangs = y
Time to carve $2x + 3y = 24$ ①
Only 10 can be decorated $x + y = 10$ ②
$2x + 2y = 20$ ③
① - ③ $y = 4$ $x = 6$
So make 4 large boomerangs
6 small boomerangs.

Tanya has used a graphical approach, but her graph of $2x + 3y = 24$ is inaccurate and should be redrawn. This graph is powerful in that it shows the entire feasible solution space - the integer points on the grid. She has not explained why her method will give the greatest profit.



$x+y \leq 10$
$2x+3y \leq 24$
5 small + 5 large = 40+50 = $90
$3y = 24-2x$
$y = 8 - \frac{2}{3}x$
Grad = $-\frac{2}{3}$
Int = 8

Follow-up lesson: individual reflection (10 minutes)

Give each student a copy of the *How Did You Work?* questionnaire. This is intended to help students reflect on the mathematical content of the task, and to reflect on the different approaches they have seen. Its important for students to have time to make notes on what they have learned, individually, so that this new knowledge will be available to them on future occasions.

> *Think carefully about your work on this task and the different methods you have seen and used. On your own, answer the review questions as carefully as you can.*

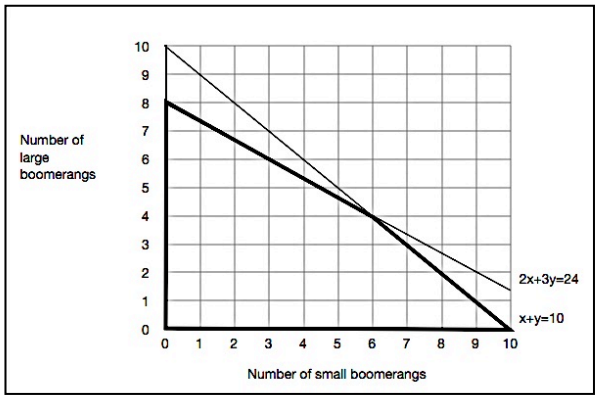
Some teachers give this as a homework task.

SOLUTIONS

Students may begin by exploring all the possible combinations of small and large boomerangs when making a total of ten boomerangs:

Number of small	Number of large	Total number	No. of carving hours	Profit made
0	10	10	30	100
1	9	10	29	98
2	8	10	28	96
3	7	10	27	94
4	6	10	26	92
5	5	10	25	90
6	4	10	24	88
7	3	10	23	86
8	2	10	22	84
9	1	10	21	82
10	0	10	20	80

The constraint on carving hours is not satisfied when more than four large boomerangs are made. This approach, however, does not include the possibility of making fewer than ten boomerangs.
A more complete approach would be to draw a graph showing all possibilities. The possible combinations to be checked are the integer points within the bold region on the graph:



Number of small	Number of large	Total number	No. of carving hours	Profit made
0	8	8	22	80
1	7	8	23	82
2	6	8	24	84
3	6	9	22	76
4	5	9	23	78
5	4	9	24	80

The maximum profit occurs, however, when six small and four large boomerangs are made.
This profit is $88. (This can be seen graphically by drawing lines of constant profit on the graph, e.g. $8x + 10y = 80$

Boomerangs

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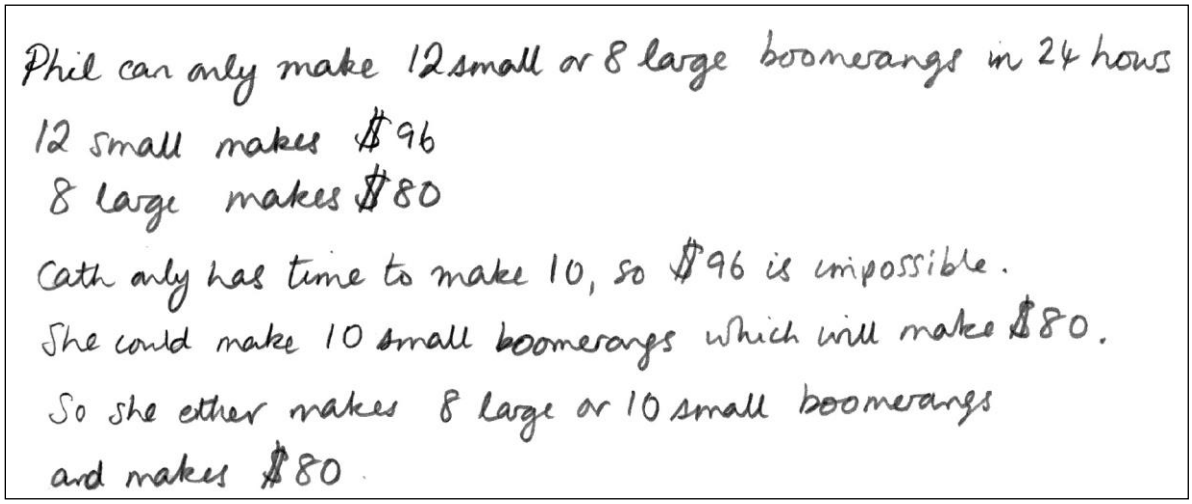
They want to make as much money for charity as they can.

How many small and large boomerangs should they make?

How much money will they then make?

Sample Responses to Discuss: Alex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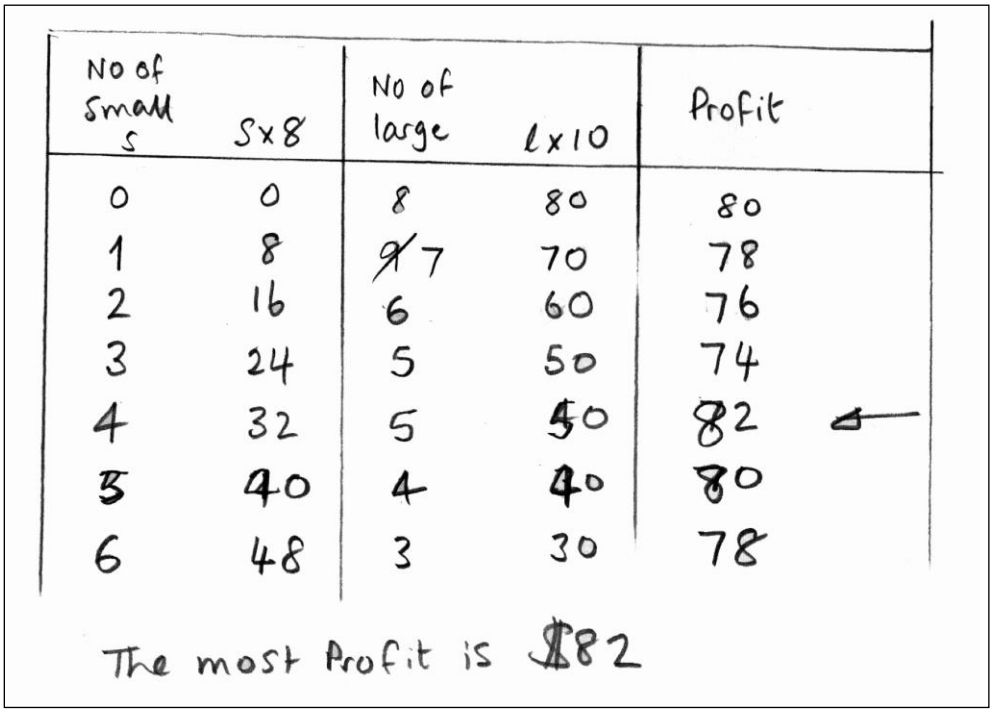
What assumptions has Alex made?

Are these assumptions correct? Explain your answer.

General comments:

Sample Responses to Discuss: Danny



No of Smalls	S x 8	No of large	L x 10	Profit
0	0	8	80	80
1	8	~~9~~ 7	70	78
2	16	6	60	76
3	24	5	50	74
4	32	5	50	82 ←
5	40	4	40	80
6	48	3	30	78

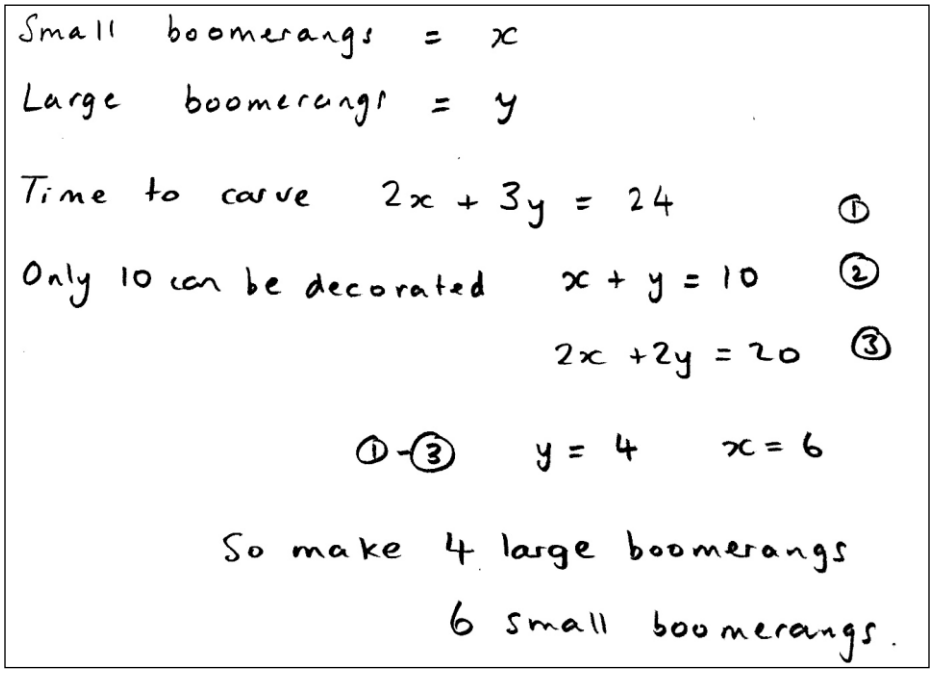
The most Profit is $82

Why do you think Danny starts with 0 small and 8 large boomerangs and stops at 6 small and 3 large boomerangs?

What piece of information has Danny forgotten to use?

General comments:

Sample Responses to Discuss: Jeremiah

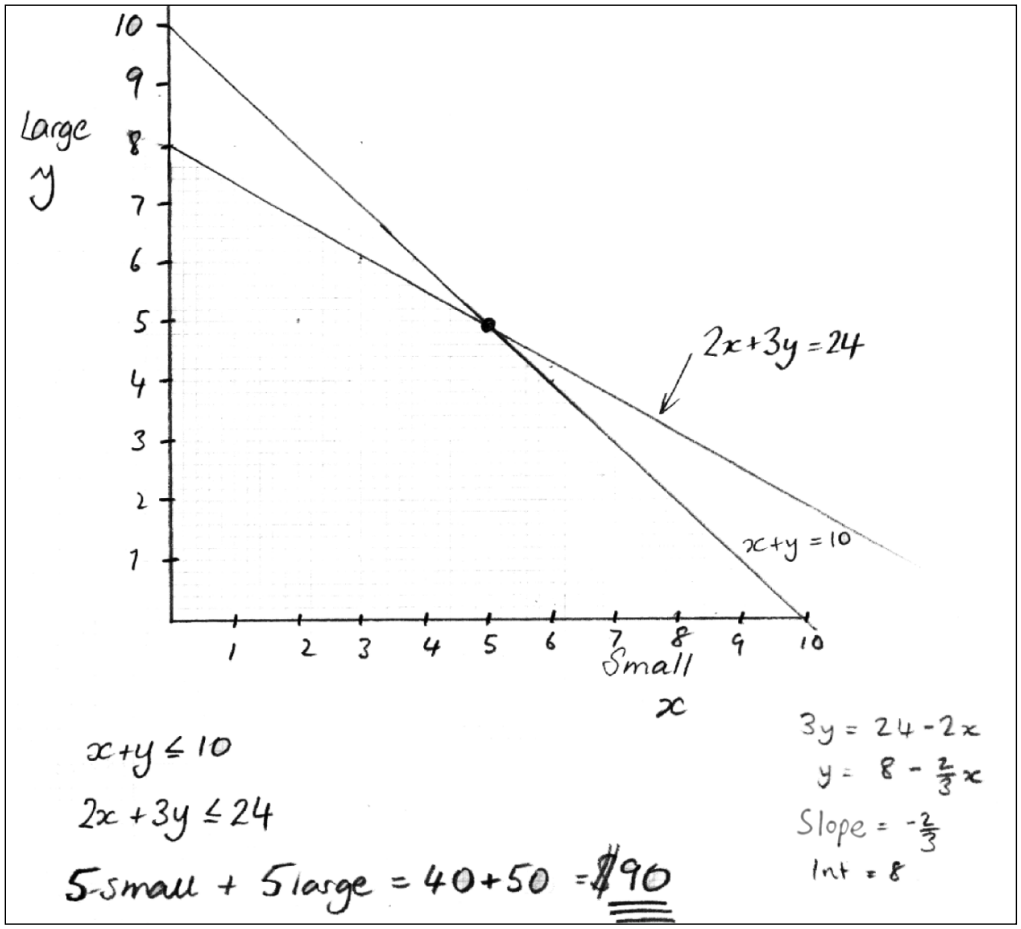


Is it correct to use the equals sign in equations 1, 2 and 3? Explain your answer?

Why is Jeremiah's solution incomplete?

General comments:

Sample Responses to Discuss: Tanya



What is the purpose of the graph?

What is the point of figuring out the slope and intercept?

General comments:

How Did You Work?

1. Whilst working on the *Boomerangs* task I could see that this problem could involve: *(Mark ☑ the most appropriate activities.)*

Calculating costs and times		*Creating equations*		*Creating inequalities*	
Drawing graphs		*Solving simultaneous equations*		*Controlling variables systematically*	

2. My own method is similar to the sample response of: ☐ OR My own method is different from **all** of the sample responses ☐

   ☐ *(insert name of sample response)*

   Because: .....

   .....

3. Our group's method is similar to the sample response of: ☐ OR Our group's method is different from **all** sample responses ☐

   ☐ *(insert name of sample response)*

   Because: .....

   .....

4. Of all the methods I have seen, the one that I prefer is: My own / Our group's / Alex's / Danny's / Jeremiah's / Tanya's / Other *(Circle)*

   Because: .....

   .....

5. What I found difficult about (a) solving this problem and (b) the mathematics in the problem was:

   (a) ..... (b) .....

   ....

   ....

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They want to make as much money for charity as they can.

How many small and large boomerangs should they make?

How much money will they then make?

Evaluating Sample Responses

- What do you like about the work?
- How has each student organized the work?
- What mistakes have been made?
- What isn't clear?
- What questions would you like to ask this student?
- In what ways might the work be improved?

Sample Responses to Discuss: Alex

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So she either makes 8 large or 10 small boomerangs

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Sample Responses to Discuss: Danny

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0	0	8	80	80
1	8	~~9~~ 7	70	78
2	16	6	60	76
3	24	5	50	74
4	32	5	~~5~~ 0	~~8~~ 2 ←
~~5~~	~~4~~ 0	4	~~4~~ 0	~~8~~ 0
6	48	3	30	78

The most Profit is ~~$~~82

Sample Responses to Discuss: Jeremiah

Small boomerangs = x

Large boomerangs = y

Time to carve $2x + 3y = 24$ ①

Only 10 can be decorated $x + y = 10$ ②

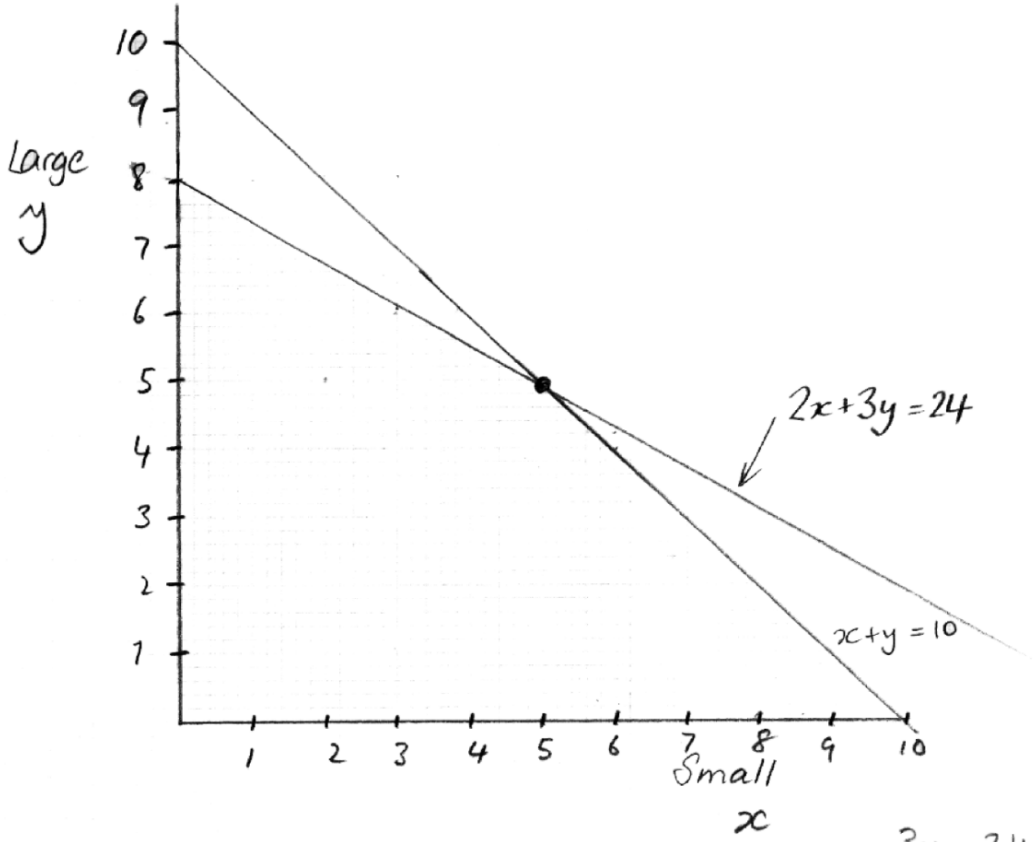
$2x + 2y = 20$ ③

① − ③ $y = 4$ $x = 6$

So make 4 large boomerangs

6 small boomerangs.

Sample Responses to Discuss: Tanya



$x + y \leq 10$

$2x + 3y \leq 24$

5 small + 5 large = 40 + 50 = $90

$3y = 24 - 2x$

$y = 8 - \frac{2}{3}x$

Slope = $-\frac{2}{3}$

Int = 8

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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We are grateful to the many teachers and students, in the UK and the US,
who took part in the classroom trials that played a critical role in developing these materials

The classroom observation teams in the US were led by
David Foster, Mary Bouck, and **Diane Schaefer**

This project was conceived and directed for
The Mathematics Assessment Resource Service (MARS) by
Alan Schoenfeld at the University of California, Berkeley, and
Hugh Burkhardt, Daniel Pead, and **Malcolm Swan** at the University of Nottingham

Thanks also to Mat Crosier, Anne Floyde, Michael Galan, Judith Mills, Nick Orchard, and Alvaro Villanueva who contributed to the design and production of these materials

This development would not have been possible without the support of
Bill & Melinda Gates Foundation

We are particularly grateful to
Carina Wong, Melissa Chabran, and **Jamie McKee**

The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org