CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Adding and Subtracting Directed Numbers

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Adding and Subtracting Directed Numbers

MATHEMATICAL GOALS

This lesson unit is intended to help students to:

- Add and subtract directed numbers (positive, negative and zero) with understanding.
- Address common misconceptions about the addition and subtraction of directed numbers.
- Explain their reasoning using diagrams.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards* for Mathematics:

6.NS: Apply and extend previous understandings of numbers to the system of rational numbers.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 6 and 7:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

This lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task designed to reveal their current understanding. You then review their responses and create questions for students to consider when improving their work.
- After a whole-class discussion that introduces the charge model for directed numbers, students work in small groups on two collaborative discussion tasks in which they match diagrams with calculations and produce missing diagrams for the remaining calculations.
- In a whole-class discussion, students discuss what they have learned.
- Finally, students revisit their initial work on the assessment task and work alone on a similar task to the introductory task.

MATERIALS REQUIRED

- Each student will need a mini-whiteboard, pen and eraser, some blank paper and copies of the assessment tasks *Directed Numbers* and *Directed Numbers (revisited)*. Some small plastic counters of two contrasting colors might be useful if readily available. Glue would be useful but is not essential.
- Each small group of students will need a copy of the two sheets *Calculations (1)* and *Calculations (2)* and cut-up copies of both *Card Sets*. The *Calculations* sheets and the *Card Sets* should all be enlarged (by the same amount), if possible, to make the cards easier to handle.

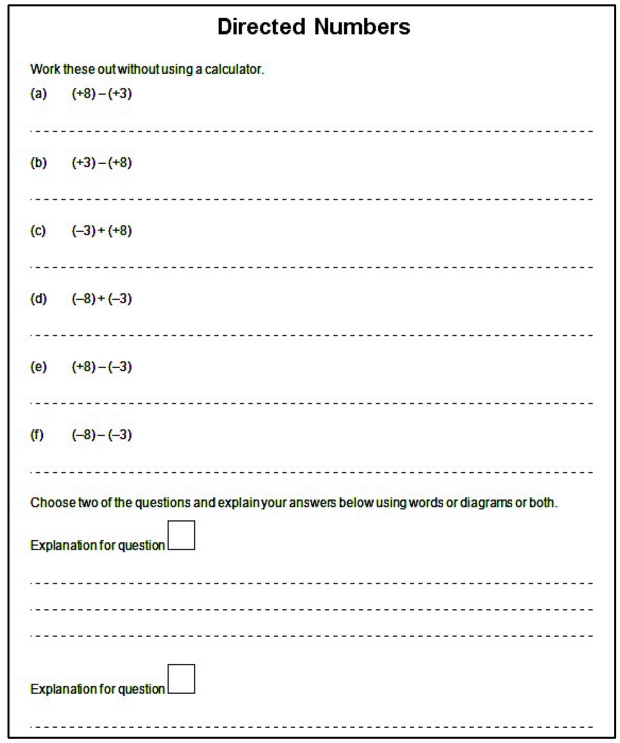
TIME NEEDED

15 minutes before the lesson, a 90-minute lesson (or two 50-minute lessons), and 15 minutes in a follow-up lesson. Exact timings will depend on the needs of your students.

BEFORE THE LESSON

Assessment task: *Directed Numbers* (15 minutes)

Have the students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.


Directed Numbers

Work these out without using a calculator.

(a) (+8) – (+3)

(b) (+3) – (+8)

(c) (–3) + (+8)

(d) (–8) + (–3)

(e) (+8) – (–3)

(f) (–8) – (–3)

Choose two of the questions and explain your answers below using words or diagrams or both.

Explanation for question

Explanation for question

Give each student a copy of the assessment task *Directed Numbers* and briefly introduce the task:

> *Write down the answers to the six calculations given, without using a calculator.*
>
> *Then choose two of your calculations to explain underneath. You can explain them in words or use diagrams, or both.*
>
> *I want to see how you think about these calculations.*

It is important that, as far as possible, students are allowed to answer the questions without assistance. They should not have access to calculators.

Students should not worry too much if they cannot understand nor do everything, because in the next lesson they will work on some tasks that should help them. Explain to students that by the end of the next lesson they should be able to answer questions such as these, with explanation, confidently. This is their goal.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem-solving approaches.

We suggest that you do not score students' work. Research shows that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions, and highlight appropriate questions for each student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students in the follow-up lesson.

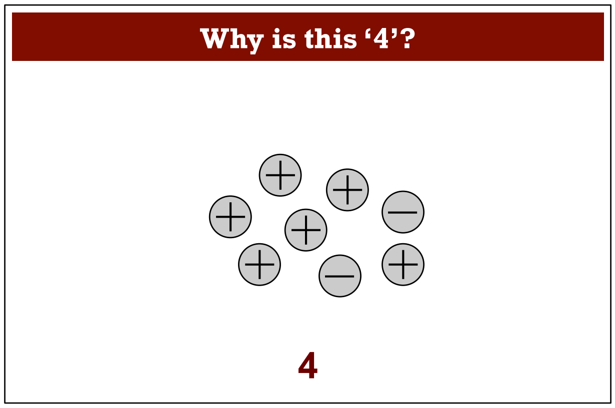
Common issues	Suggested questions and prompts
Interprets the brackets as multiplication For example: The student writes '24' or '-24' as the answer to (a).	• What is the answer to (+8) x (+3)?
Confuses the sign of the numbers with the operation For example: The student writes '11' as the answer to (a).	• What is the answer to (+8) + (+3)? • Do you think you should obtain the same answer for (a) and (+8) + (+3)?
Subtracts the wrong way round in order to obtain a positive answer For example: The student writes '5' or '+5' as the answer to (b).	• Can you explain the difference between the calculations in parts (a) and (b)? • Do you think you should obtain the same answer for (a) and (b)?
Interprets 'addition' to a negative number as subtraction For example: The student writes '-11' as the answer to (c) or '-5' as the answer to (d).	• What is the answer to (-3) - (+8)?
Applies 'two minuses make a plus' inappropriately For example: The student writes '11' or '+11' as the answer to (d) or '5' or '+5' as the answer to (f).	• What is the answer to (+8) + (+3)?
Interprets subtraction of a negative number as subtraction of a positive number. For example: The student writes '5' or '+5' as the answer to (e).	• What is the answer to (+8) + (-3)?
Does not offer an explanation For example: The student leaves the bottom section of the sheet empty.	• How did you work out the answers? • How would you convince someone else that your answers are correct?
Uses number lines or writes about 'journeys' backwards/forwards or up/down For example: The student says for (a) you have to 'go backwards 3 units' or 'go down 3 feet'.	• Can you explain your answers in any other ways?
Uses a charge model or makes reference to ideas of debit/credit or temperature For example: The student says for (a) the temperature gets 'three degrees cooler' or that somebody 'gives away three dollars'.	• Can you explain your answers in any other ways?

SUGGESTED LESSON OUTLINE

Whole-class introduction (20 minutes)

Do not return the pre-assessment work at the beginning of the lesson. Students will receive this in the follow-up lesson.

Begin by displaying Slide P-1.



Can you explain why this diagram is showing 4?

Students may be confused if they have not encountered anything like this before, as there are 8 objects. If no one has any idea, you could ask:

Can you describe what you see in the picture?

Someone will mention words such as ‘plus’, ‘positive’, ‘minus’ and ‘negative’. If the students are really stuck you could cover with your hand everything except the right-hand pair of one negative and one positive and ask the class:

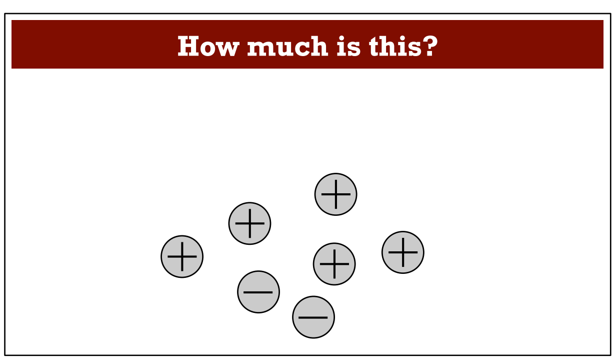
How much is there here?

They might say ‘two’, but someone will realize that the answer is ‘nothing’. Returning to the original question (by removing your hand), students may comment on ‘a plus and a minus canceling each other out’ or on there being four more ‘pluses’ than ‘minuses’.

When a positive charge and a negative charge cancel each other out, this corresponds to the fact that $(+1) + (-1) = 0$. You could illustrate this by crossing out a pair of opposite charges, although some students might find crossing out hard to understand.

You don’t need to wait for everyone to grasp the idea – as soon as a few seem to understand, move on to the next slide, as the model will become clearer with more examples.

Now show Slide P-2:



Slide P-3 shows the answer, which is 3. (If students say ‘plus three’ or ‘positive 3’, that is fine.)

Now show Slide P-4.

Making 3

Show me three more ways of making 3.

Can you do one using 11 charges altogether?

On your mini-whiteboards, draw me three ways like this of making 3.

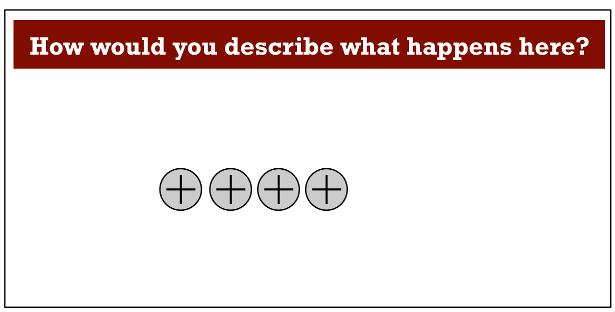
If you have small plastic counters of two contrasting colors, students might find these useful to use as they employ trial and error to find possible answers.

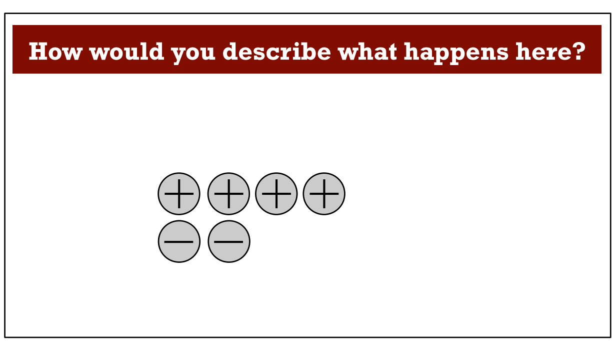
Any answer involving n positives and $n - 3$ negatives is correct ($n > 3$). Students might enjoy discovering that they can use very large numbers of symbols to make a relatively small number like 3.

To use 11 symbols altogether, students will need 7 positives and 4 negatives, because if $2n - 3 = 11$ then $n = 7$. (They are not expected to solve this algebraically, but the algebra does show that any odd total number of charges is possible.)

Slides P-5 to P-10 are intended to help students see the need for placing a + or – before each number and you should stress the importance of this.

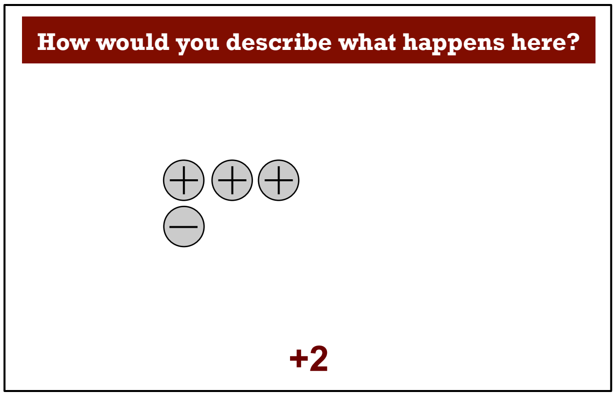
Then show Slide P-11 followed by P-12:

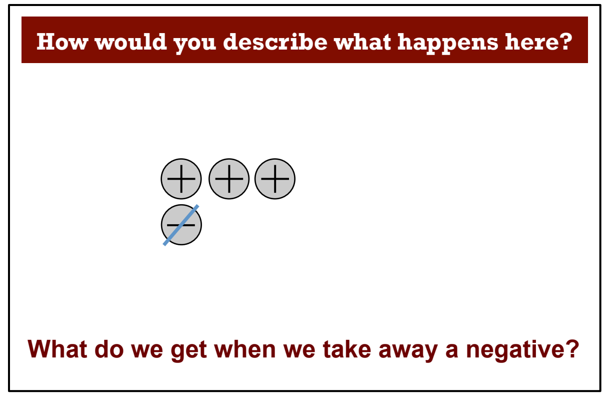




Slide P-11 shows (+4) and then two negatives are added in Slide P-12. Slides P-13 to P-15 formalize this as $(+4) + (-2) = (+2)$. Invite several students to explain the connection between the picture and the calculation in their own words.

Now show Slides P-16 and P-17:





What does the crossing out take away? How would we write that?
[Positive two 'take away' negative one leaves positive three.]

Slides P-18 to P-20 develop symbols for this: (+2)-(-1)=(+3)

This is difficult and may be worth allowing students time to explain to each other in pairs.

Explain the story that this slide represents.

Slides P-21 to P-23 offer a further example. If not everyone understands the subtraction of negatives at this stage, that is fine, as the first collaborative task does not involve this and students will have opportunity later to think about this more:

If you are a bit confused about these positives and negatives at the moment, don't worry. You will learn more about this in the group tasks that we are going to do.

Collaborative small-group work 1 (20 minutes)

Ask students to work in groups of two or three. Give each group some blank paper, a copy of *Calculations (1)*, and the three cut-out cards from *Card Set 1* only. If you have small plastic counters in two contrasting colors, students might also find these useful to move around on the table as they think about this task.

I want you to decide where the three cards go on the grid.

Fill in the answers to those three calculations after the "equals" signs.

Make sure that your answers match what you see in the pictures.

Then you need to make up drawings to go with the other three calculations and write in the answers to those calculations as well.

Explain how students are to work together, using Slide P-24:

Working Together 1

1. Choose one of the three cards and match it with one of the calculations.
2. Write the answer to the calculation after the equals sign.
3. Check that the answer matches what you see in the drawing.
4. Do the same with the other two cards.
5. Make up drawings to fit the remaining three calculations and write down the answers to those three calculations.
6. The answers should match the drawings.
7. Everyone in the group should agree.

Students may use their blank paper for rough calculations and to explain their thinking to each other. They should not use calculators.

Confident students may want to go through writing in all the answers before thinking about the diagrams and cards. There is no harm in them doing that, but you could ask them how they know whether their answers are correct or incorrect. They will need to use the cards and make drawings afterwards in order to convince someone else that they are right and in the process they may realize that they have made some mistakes and find that they need to change some of their answers.

If they are sure that they will not need to make any changes, then students could glue down their answers if they wish.

Extending the lesson over two days

If you are taking two days to complete the unit then you may want to end the first lesson here, ensuring that students have glued down their answers. Then, at the start of the second day, students can reflect on their work before attempting a similar collaborative task.

Whole-class discussion (15 minutes)

If students carry out the previous collaborative task quickly and confidently, you might choose to omit this discussion and go straight on to the second collaborative task.

Which calculations did you find easiest/hardest? Why?

What drawing did you make for this calculation? Why?

Can you explain what your drawing shows?

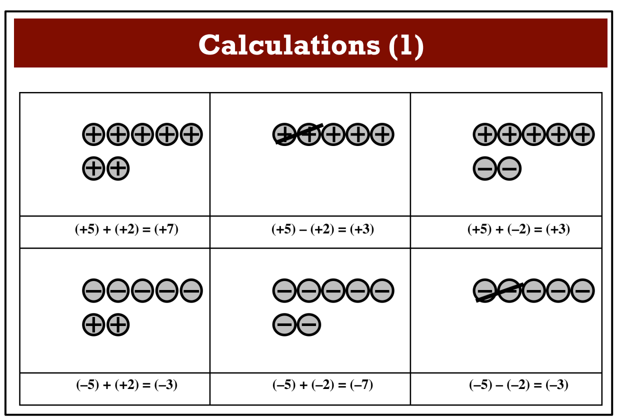
Did anyone else do the same or something different?

Which drawing do we prefer? Why?

What answer did you obtain for this calculation? Why?

Did anyone obtain a different answer? Why?

Slide P-25 shows the completed *Calculations (1)* sheet with the correct answers and possible diagrams, although slightly different diagrams would also be correct:

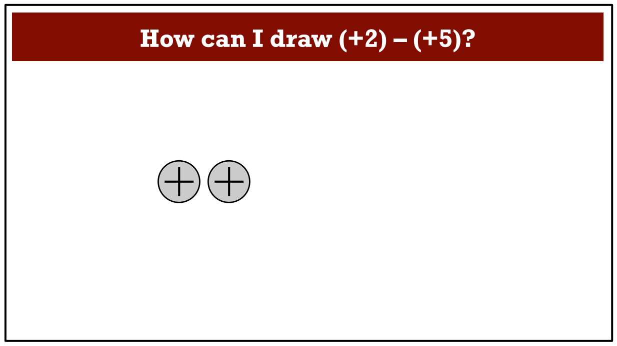


You could ask students what they notice from the finished sheet. For example:

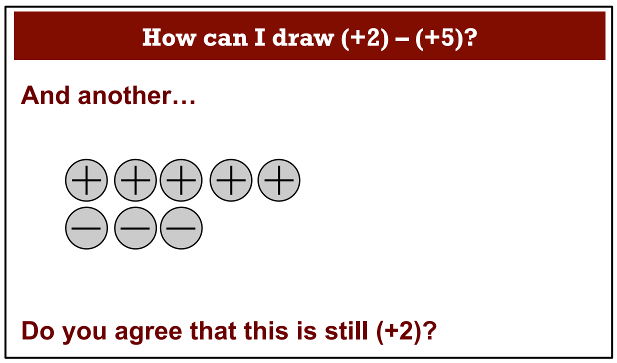
Is there a difference between the calculations (+5) – (+2) and (+5) + (-2)?

These calculations give the same answer but have different diagrams and students could talk about how they envisage the two processes differently. They may also comment on the fact that two of the other calculations give the same answer and talk about why. There are interesting patterns here to discuss.

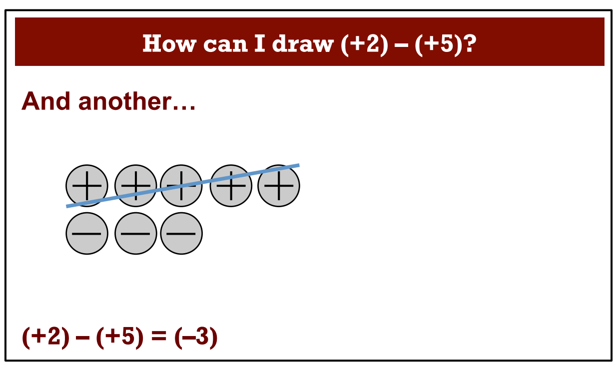
Now show Slide P-26, which poses a more difficult calculation, where the thing that is being subtracted does not appear to be present:



Slides P-27 to P-29 show how other ways of writing (+2) are more useful here:



Then Slides P-30 to P-32 show how (+5) can now be subtracted:



Students will have opportunities to think more about this approach in the second collaborative activity.

Collaborative small-group work 2 (20 minutes)

Ask students to return to their groups of two or three. Give each group some blank paper, a copy of *Calculations (2)*, and the three cut-up cards from *Card Set 2*. If you have small plastic counters in two contrasting colors, students might also find these useful to move around on the table as they think about this work.

Now I want you to see whether you can do the harder subtraction calculations using, when you need to, diagrams like the one you have just seen.

Go through the same process as before with the Calculations (2) sheet.

Remind students how you would like them to work together, using Slide P-33:

Working Together 2

1. Choose one of the three cards and match it with one of the calculations.
2. Write the answer to the calculation after the equals sign.
3. Check that the answer matches what you see in the drawing.
4. Do the same with the other two cards.
5. Make up drawings to fit the remaining three calculations and write down the answers to those three calculations.
6. The answers should match the drawings.
7. Everyone in the group should agree.

Again, students may use their blank paper for rough thinking and to explain their thinking to each other. They should not use calculators.

While students are working you have two tasks: to note different student approaches to the task and to support student problem solving.

Note different student approaches

Listen to and watch students carefully. Notice how students make a start on the task, where they get stuck, and how they overcome any difficulties. Which calculations do they find it easiest/hardest to make diagrams for? Which do they find easiest/hardest to interpret to find the answer to the calculation? Where are their explanations strong/weak? What misconceptions are manifest? What disagreements are common?

In particular, notice whether students are addressing the difficulties they experienced in the assessment task. Also note any common mistakes. For example, some students might think that ‘two minuses make a plus’ means that $\ominus\ominus$ is the same as $\oplus$, which is wrong. Two negatives don’t cancel out – it is a negative and a positive that cancel out.

You might wish to make a distinction in your use of language between the words ‘positive’ and ‘negative’ as adjectives relating to a single number and ‘plus’ and ‘minus’ as verbs relating to operations; e.g., ‘positive 3 minus negative 2 equals positive 5’.

Support student problem solving

Help students to work constructively together. Remind them to look at the slide for instructions on how to work. Check that students listen to each other and encourage them to show their ideas with counters or by doing rough drawings on their blank paper.

Try not to solve students’ problems or do the reasoning for them. Instead, you might offer strategic advice or ask strategic questions to suggest ways of moving forward:

> *If you’re stuck with that card, you could put it to one side and place one of the others first.*
>
> *What do you think the answer is going to be to the calculation? Does that help you to make the diagram?*
>
> *If you need to subtract (+2) then you need to have a (+2) to start with. But you still need to start with a total of (-5), so you need to make a picture that has (+2) in it but is equal to (-5) altogether. How can you draw that?*

If a group of students finish placing all the cards and completing all the blank spaces, ask them to choose two different numbers (positive integers less than 10) and make a new table showing all the possible addition and subtraction calculations if either number could be positive or negative. (There should be 16 of these calculations.) They could then try to work out the answers, using diagrams when they are unsure.

Whole-class discussion (15 minutes)

Conduct a whole-class discussion about what has been learned and explore the different diagrams that have been drawn. Have you noticed some interesting misconceptions as you circulated among the groups? If so, you may want to focus the discussion on these.

Which calculations were easiest/hardest this time? Why?

What drawing did you make for this calculation? Why?

Can you explain in words what your drawing shows?

Does anyone have a different way of explaining it?

Did anyone else do the same or something different?

Which drawing do you prefer? Why?

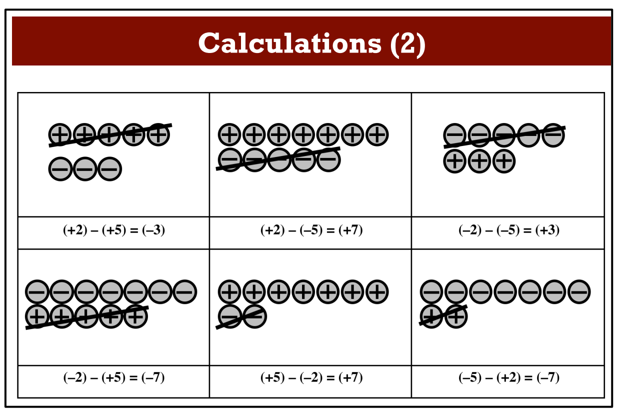
Who agrees/disagrees? Why?

What answer did you obtain for this calculation? Why?

Did anyone obtain a different answer? Why?

What do other people think?

Slide P-34 shows the completed *Calculations (2)* sheet with the correct answers and possible diagrams, although, again, slightly different diagrams would also be correct:



You could ask students to compare calculations from the finished sheet. For example:

There are two pairs of calculations with the same answer. Can you explain why this happens?

What else is the same about some of the diagrams? What is different?

There are interesting patterns here to discuss. Students might comment on the fact that each time all of one sign of charge are crossed out (i.e. all of the positives or all of the negatives). This is efficient but is not the only possibility. Any number of additional pairs of positives and negatives would also give the correct result.

If the class has managed the task successfully you could use mini-whiteboards to ask questions involving larger numbers. Where there are too many charges to draw them all, can students nevertheless use what they have learned to visualize the answer? This might be a next step for some.

Reviewing the assessment task (15 minutes)

Give each student a copy of the review task, *Directed Numbers (revisited)* and their original scripts from the assessment task, *Directed Numbers*. If you have not added questions to individual pieces of work then write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

> *Look at your original responses and the questions [on the board/written on your script]. Answer these questions and revise your response.*
>
> *Now look at the new task sheet, Directed Numbers (revisited). Can you use what you have learned to answer these questions?*

Some teachers give this as homework.

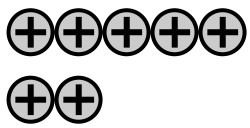
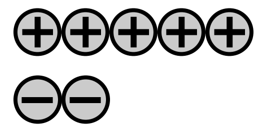
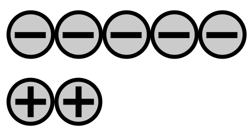
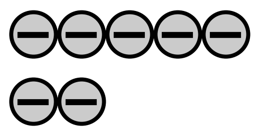
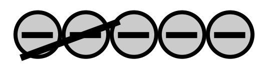
SOLUTIONS

Assessment task: *Directed Numbers*

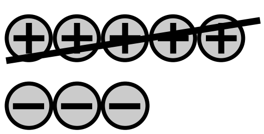
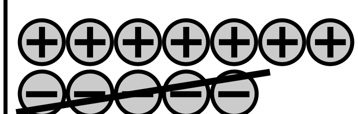
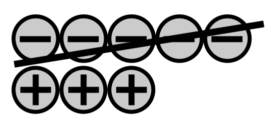
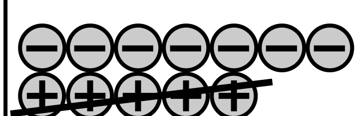
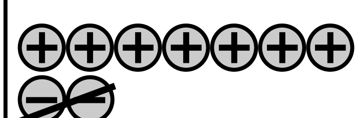
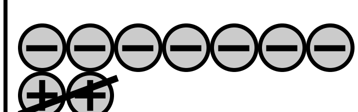
(a) +5; (b) -5; (c) +5; (d) -11; (e) +11, (f) -5.

Collaborative tasks

Calculations (1)

		
(+5) + (+2) = (+7)	**(+5) – (+2) = (+3)**	**(+5) + (–2) = (+3)**
		
(–5) + (+2) = (–3)	**(–5) + (–2) = (–7)**	**(–5) – (–2) = (–3)**

Calculations (2)

		
(+2) – (+5) = (–3)	**(+2) – (–5) = (+7)**	**(–2) – (–5) = (+3)**
		
(–2) – (+5) = (–7)	**(+5) – (–2) = (+7)**	**(–5) – (+2) = (–7)**

Assessment task: *Directed Numbers (revisited)*

(a) +2; (b) -2; (c) +2; (d) -10; (e) +10; (f) -2.

Directed Numbers

Work these out without using a calculator.

(a) (+8) – (+3)

(b) (+3) – (+8)

(c) (–3) + (+8)

(d) (–8) + (–3)

(e) (+8) – (–3)

(f) (–8) – (–3)

Choose two of the questions and explain your answers below using words or diagrams or both.

Explanation for question $\boxed{}$

Explanation for question $\boxed{}$

Calculations (1)

$(+5) + (+2) =$	$(+5) - (+2) =$	$(+5) + (-2) =$
$(-5) + (+2) =$	$(-5) + (-2) =$	$(-5) - (-2) =$

DO NOT CUT OUT ANYTHING ON THIS PAGE

Calculations (2)

$(+2) - (+5) =$	$(+2) - (-5) =$	$(-2) - (-5) =$
$(-2) - (+5) =$	$(+5) - (-2) =$	$(-5) - (+2) =$

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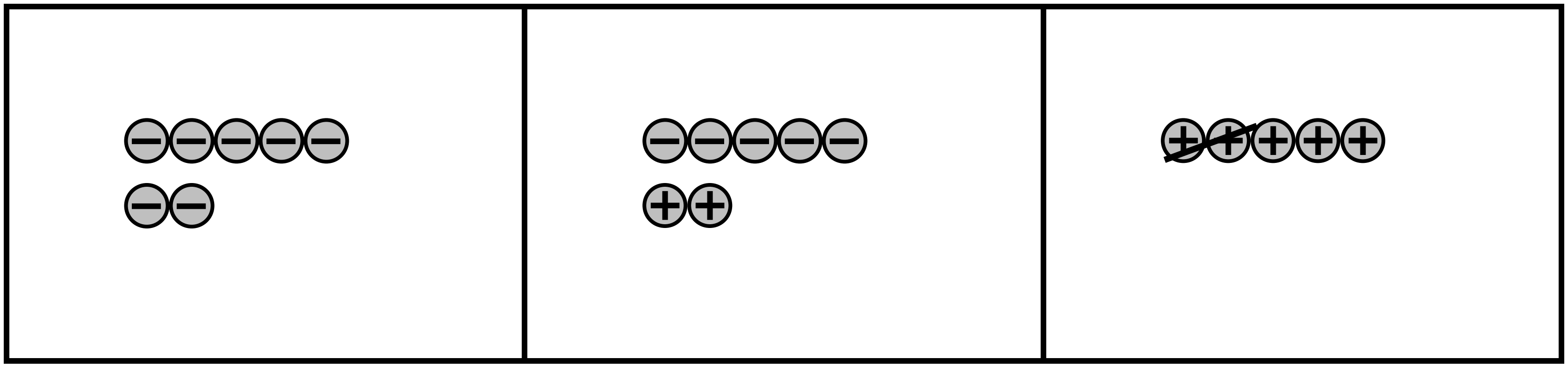
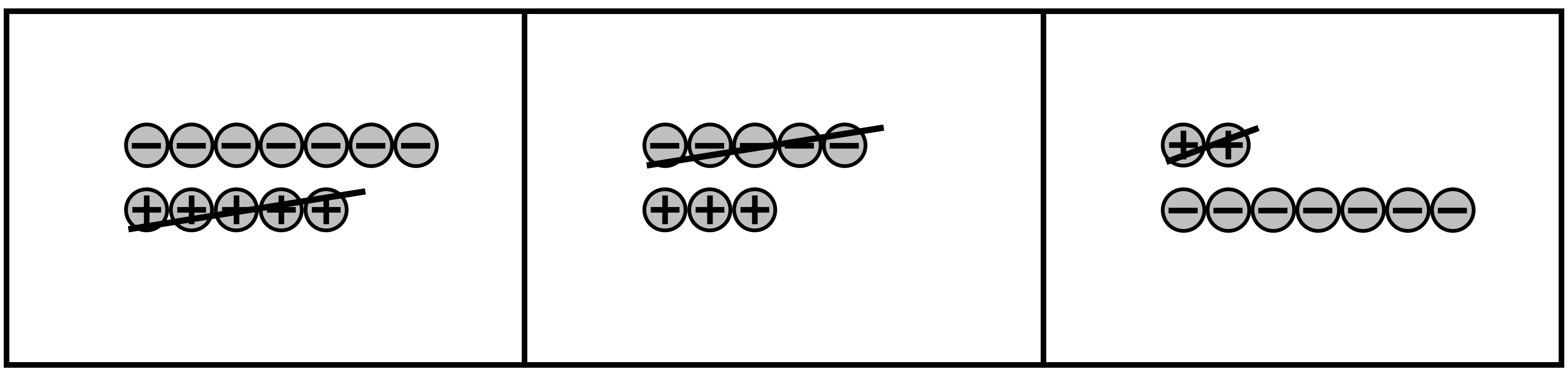
Card Set (1)



CUT OUT THESE THREE CARDS FOR 'CALCULATIONS (1)'

Card Set (2)



THEN CUT OUT THESE THREE CARDS FOR 'CALCULATIONS (2)'

Directed Numbers (revisited)

Work these out without using a calculator.

(a) (+6) – (+4)

(b) (+4) – (+6)

(c) (–4) + (+6)

(d) (–6) + (–4)

(e) (+6) – (–4)

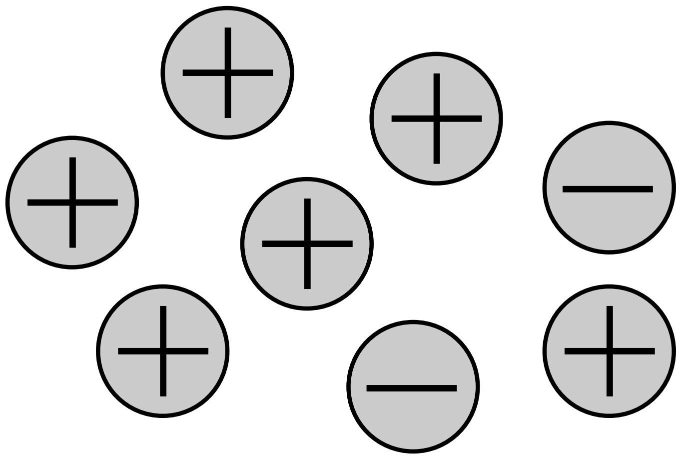
(f) (–6) – (–4)

Choose two of the questions and explain your answers below using words or diagrams or both.

Explanation for question $\square$

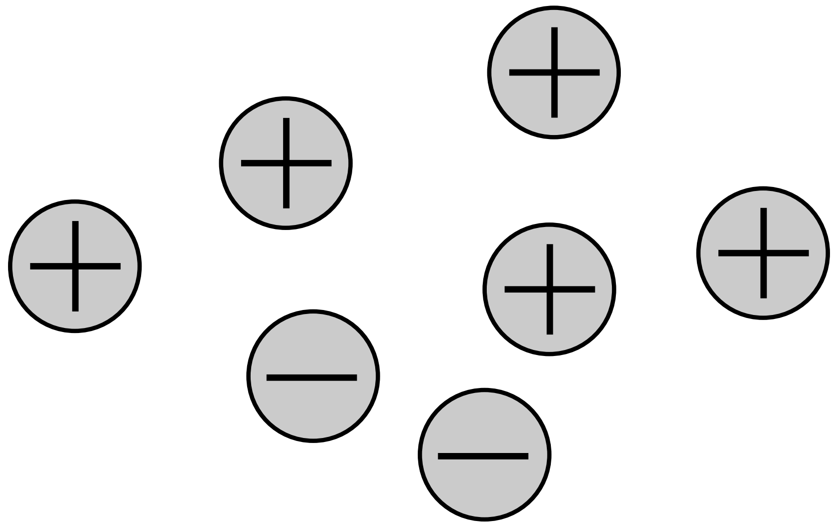
Explanation for question $\square$

Why is this ‘4’?

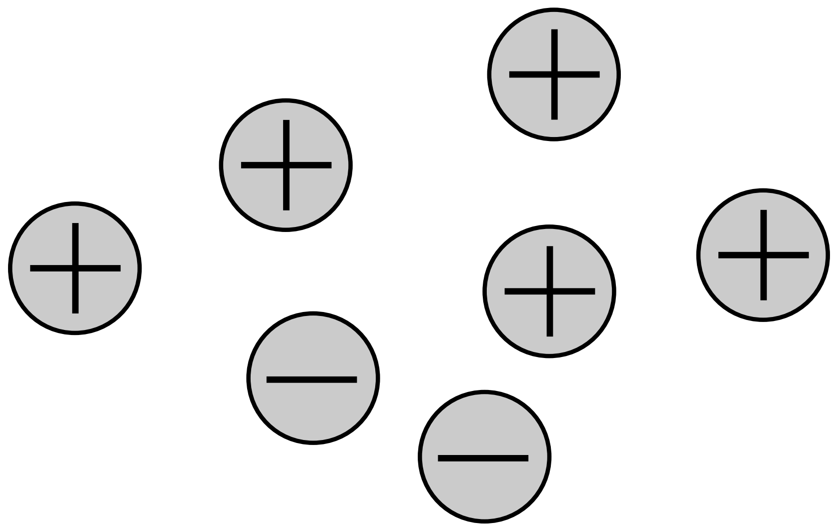


4

How much is this?



How much is this?



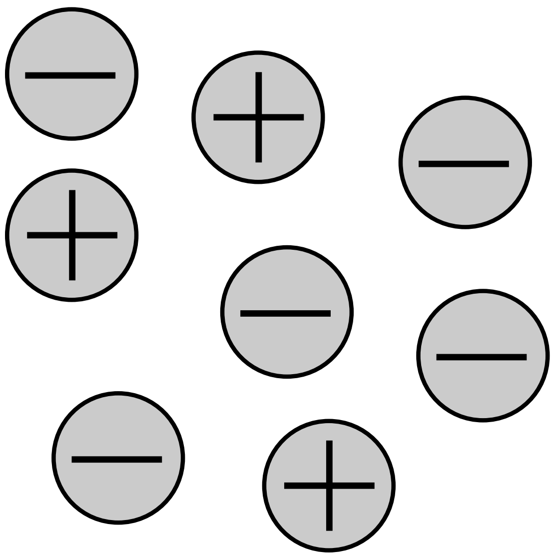
3

Making 3

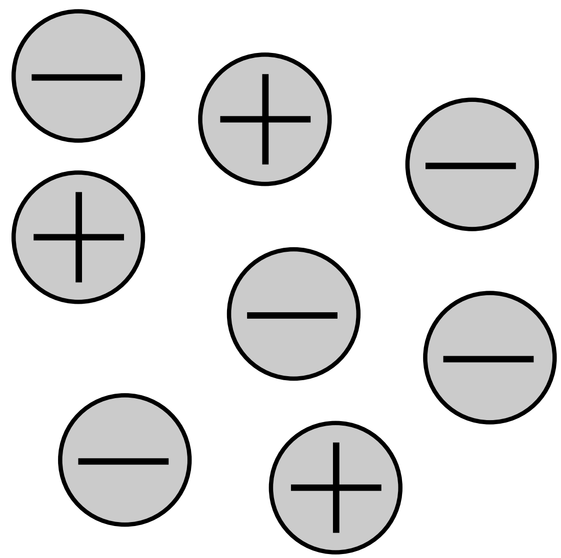
Show me three more ways of making 3.

Can you do one using 11 charges altogether?

How much is this?

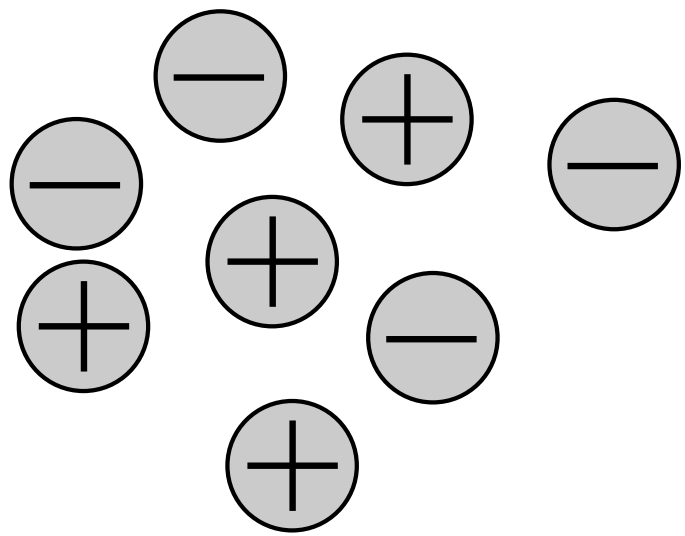


How much is this?

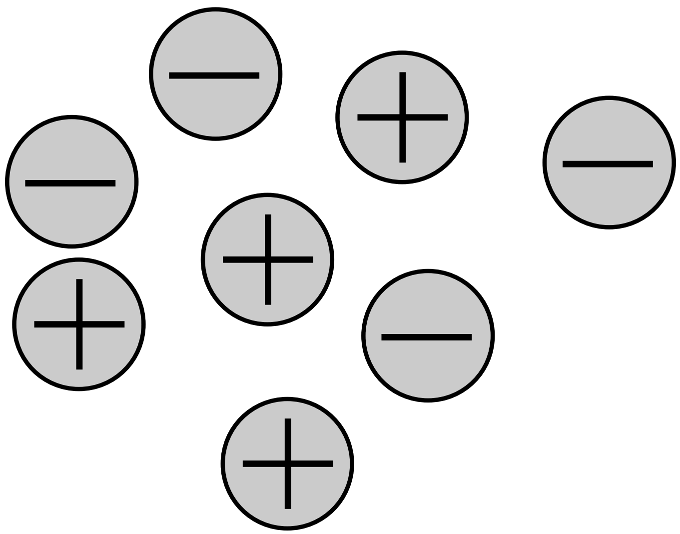


–2

How much is this?

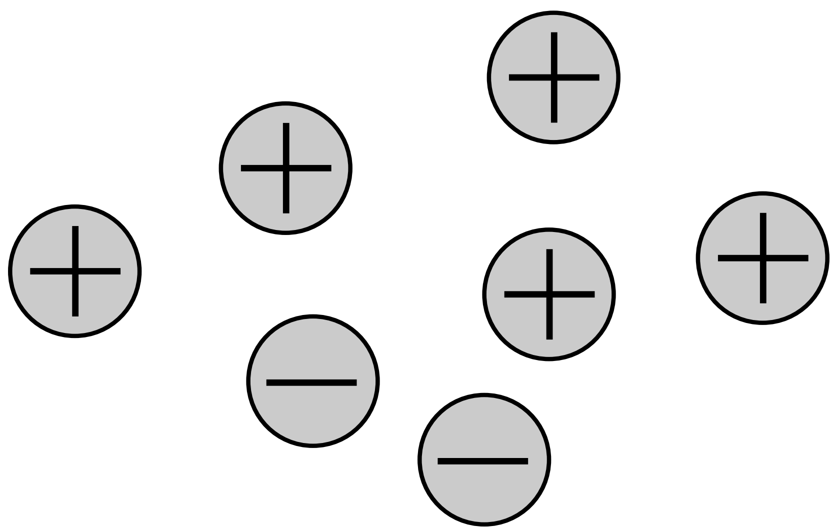


How much is this?

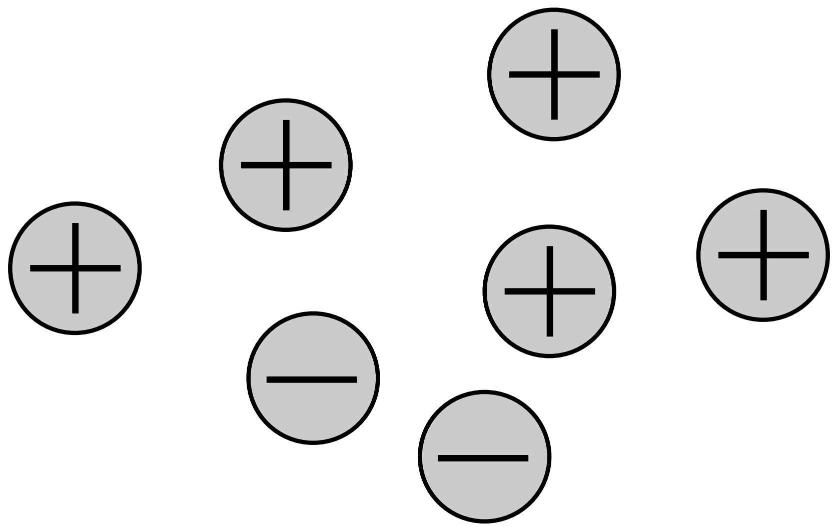


0

How much is this?

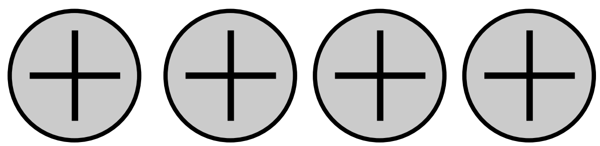


How much is this?

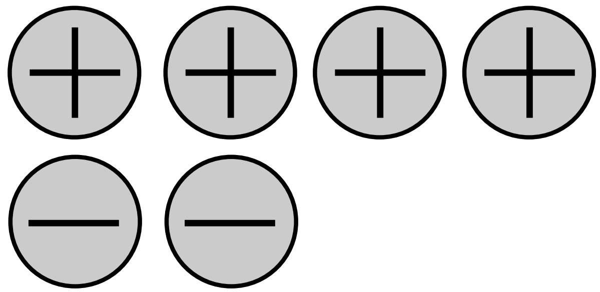


+3

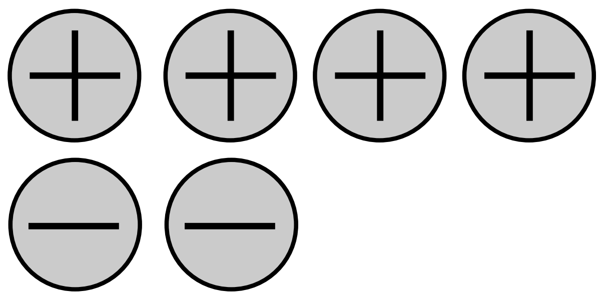
How would you describe what happens here?



How would you describe what happens here?

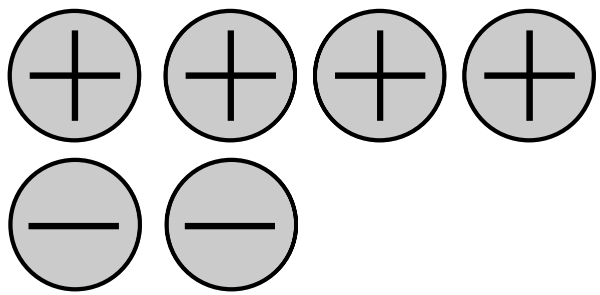


How would you describe what happens here?



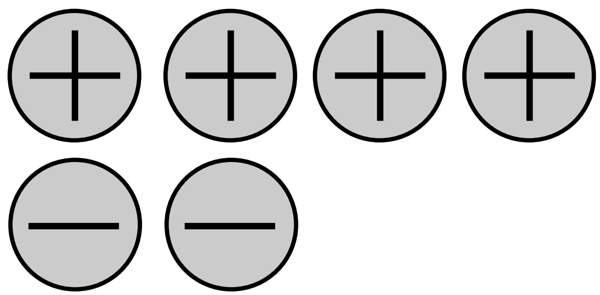
$$(+4)\ +$$

How would you describe what happens here?



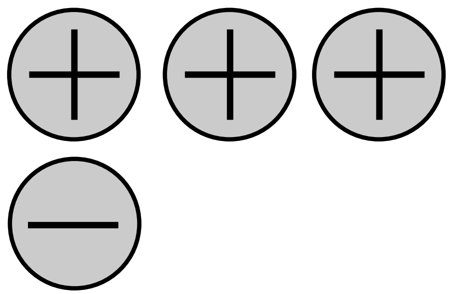
$$(+4) + (-2) =$$

How would you describe what happens here?



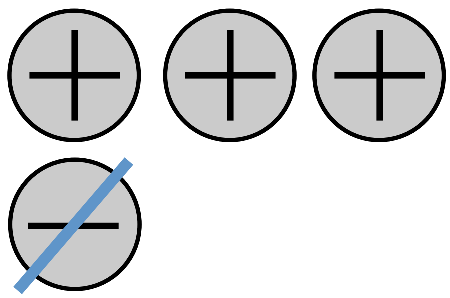
$$(+4) + (-2) = (+2)$$

How would you describe what happens here?



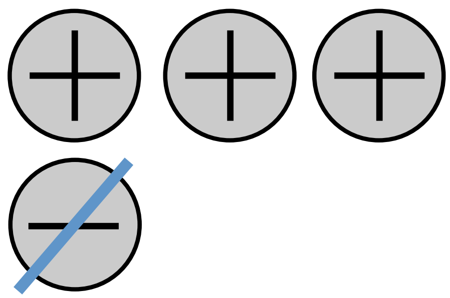
+2

How would you describe what happens here?



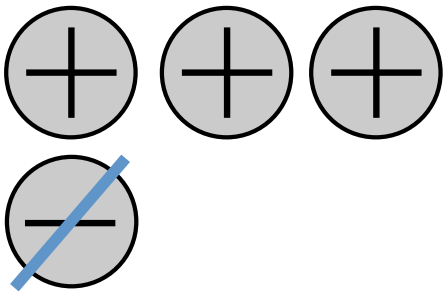
What do we get when we take away a negative?

How would you describe what happens here?



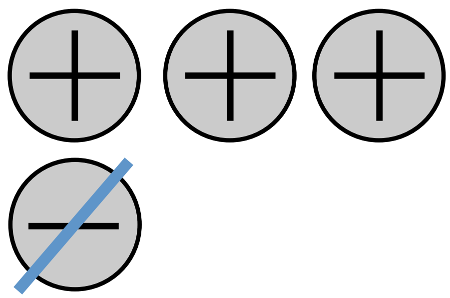
$$(+2) -$$

How would you describe what happens here?



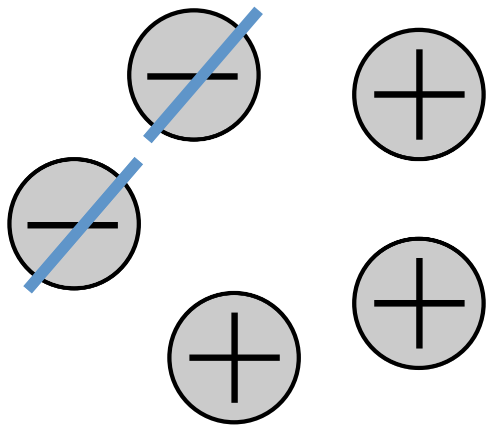
$$(+2) - (-1) =$$

How would you describe what happens here?

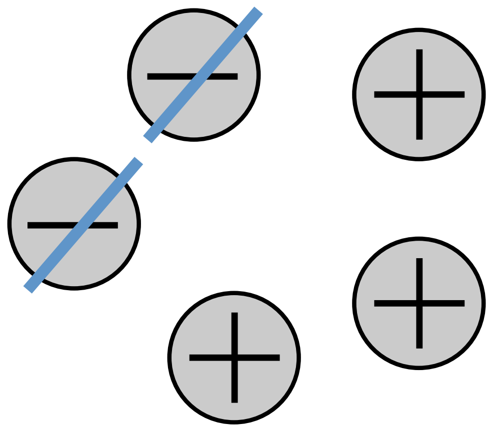


$$(+2) - (-1) = (+3)$$

What's the story here?

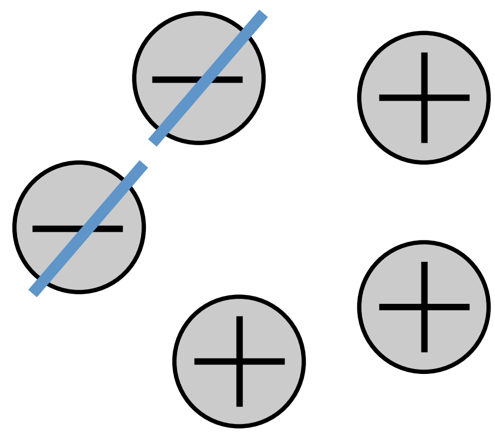


What's the answer?



$$1 - (-2) =$$

What's the answer?

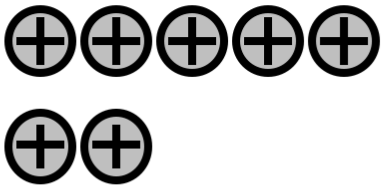
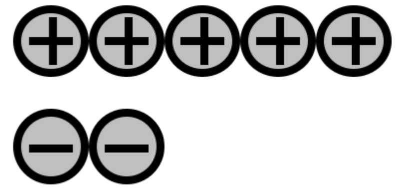
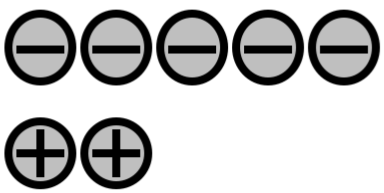
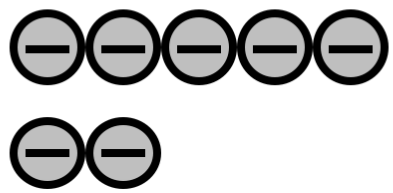


$$1 - (-2) = 3$$

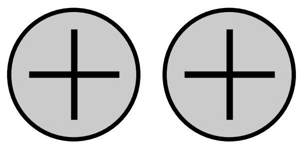
Working Together 1

1. Choose one of the three cards and match it with one of the calculations.
2. Write the answer to the calculation after the equals sign.
3. Check that the answer matches what you see in the drawing.
4. Do the same with the other two cards.
5. Make up drawings to fit the remaining three calculations and write down the answers to those three calculations.
6. The answers should match the drawings.
7. Everyone in the group should agree.

Calculations (1)

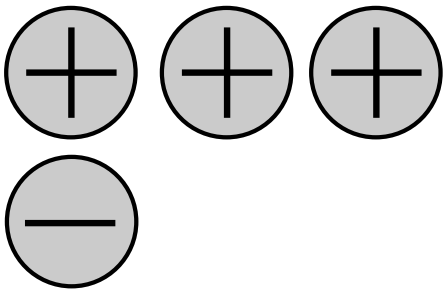
		
(+5) + (+2) = (+7)	(+5) – (+2) = (+3)	(+5) + (–2) = (+3)
		
(–5) + (+2) = (–3)	(–5) + (–2) = (–7)	(–5) – (–2) = (–3)

How can I draw (+2) – (+5)?



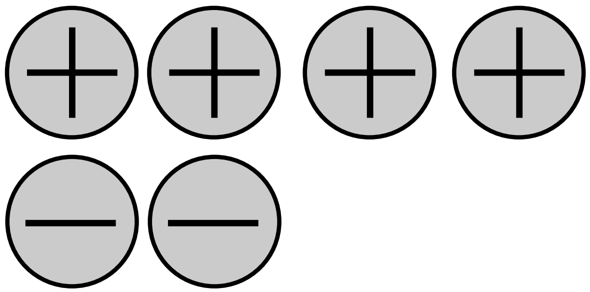
How can I draw (+2) – (+5)?

Another way of thinking of (+2)…



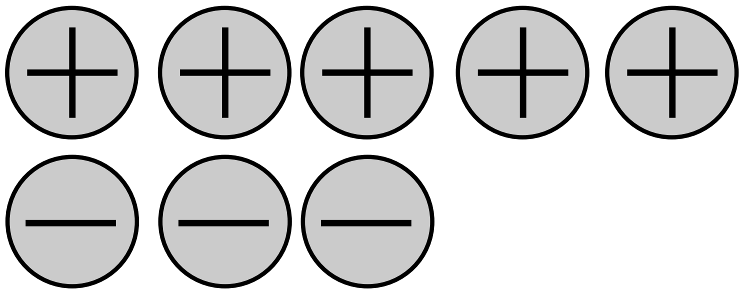
How can I draw (+2) – (+5)?

And another…



How can I draw (+2) – (+5)?

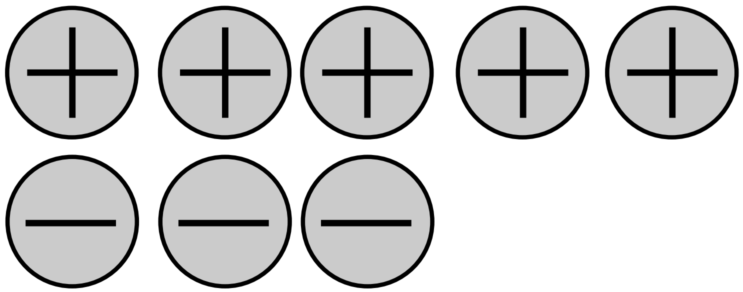
And another…



Do you agree that this is still (+2)?

How can I draw (+2) – (+5)?

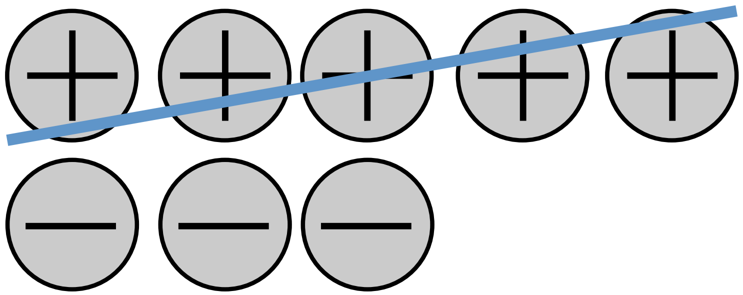
And another…



How can I take away (+5)?

How can I draw (+2) – (+5)?

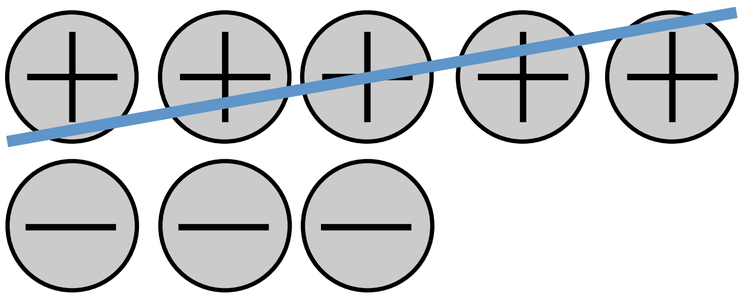
And another…



(+2) – (+5) =

How can I draw (+2) – (+5)?

And another…



(+2) – (+5) = (–3)

Working Together 2

1. Choose one of the three cards and match it with one of the calculations.

2. Write the answer to the calculation after the equals sign.

3. Check that the answer matches what you see in the drawing.

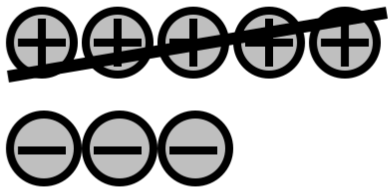
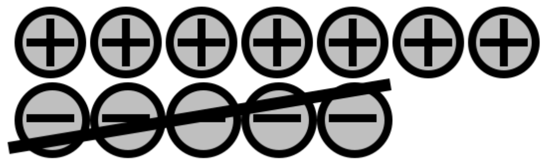
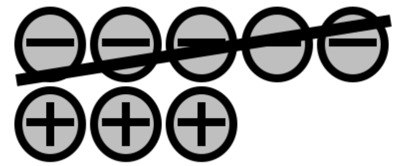
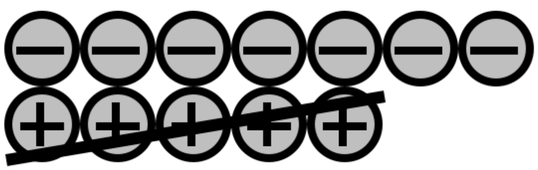
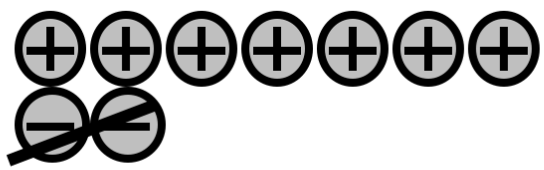
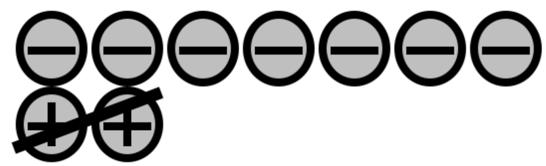
4. Do the same with the other two cards.

5. Make up drawings to fit the remaining three calculations and write down the answers to those three calculations.

6. The answers should match the drawings.

7. Everyone in the group should agree.

Calculations (2)

		
(+2) – (+5) = (–3)	(+2) – (–5) = (+7)	(–2) – (–5) = (+3)
		
(–2) – (+5) = (–7)	(+5) – (–2) = (+7)	(–5) – (+2) = (–7)

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
Shell Center Team at the Center for Research in Mathematical Education
University of Nottingham, England:

Malcolm Swan,
Nichola Clarke, Clare Dawson, Sheila Evans, Colin Foster, and **Marie Joubert**
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and **Daniel Pead**

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The classroom observation teams in the US were led by
David Foster, Mary Bouck, and **Diane Schaefer**

This project was conceived and directed for
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Hugh Burkhardt, Daniel Pead, and **Malcolm Swan** at the University of Nottingham

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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org