## CONCEPT DEVELOPMENT



Mathematics Assessment Resource Service
University of Nottingham \& UC Berkeley

## Representing Variability with Mean, Median, Mode, and Range

## MATHEIMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Calculate the mean, median, mode, and range from a frequency chart.
- Use a frequency chart to describe a possible data set, given information on the mean, median, mode, and range.


## COMMMON CORE STATE STANDARDS

This lesson relates to the following Standards for Mathematical Content in the Common Core State Standards for Mathematics:
6.SP: Develop understanding of statistical variability.

Summarize and describe distributions.
This lesson also relates to all the Standards for Mathematical Practice in the Common Core State Standards for Mathematics, with a particular emphasis on Practices 1, 2, 3, and 7:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

## INTRODUCTION

This lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task designed to reveal their current understanding and difficulties. You then review their responses and create questions for students to consider when improving their work.
- After a whole-class introduction, students work in small groups on a collaborative task, matching bar charts with statistical tables.
- To end the lesson there is a whole-class discussion.
- In a follow-up lesson, students again work alone on a task similar to the initial assessment task.


## MATERIALS REQUIRED

- Each student will need a copy of the two assessment tasks: Penalty Shoot-Out and Boy Bands, a mini-whiteboard, a pen, and an eraser.
- Each small group of students will need Card Set: Bar Charts, Card Set: Statistics Tables (already cut-up), a large sheet of paper, and a glue stick.
- There is a projector resource to support whole-class discussions.


## TIME NEEDED

15 minutes before the lesson, a 95-minute lesson (or two shorter lessons), and 15 minutes in a followup lesson. Timings given are only approximate. Exact timings will depend on the needs of the class.

## BEFORE THE LESSON

## Assessment task: Penalty Shoot-Out (15 minutes)

Have students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the lesson that follows.

Give each student a copy of the assessment task: Penalty Shoot-Out.

Read through the questions and try to answer them as carefully as you can.
It is important that, as far as possible, students are allowed to answer the questions without your assistance.

Students should not worry too much if they cannot understand or do everything, because in the next lesson they will work on a similar task, which should help them. Explain to students that by the end of the next lesson, they should be able to answer questions such as these confidently. This is their goal.

## Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches.

We suggest that you do not score students' work. The research shows that this will be counter-productive, as it will encourage


Show all your work. students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the Common issues table on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students in the follow-up lesson.

Common issues

## Suggested questions and prompts

## Misinterprets the axes on the bar chart

For example: The student states that there were six people involved in the shoot-out (Q1a).

Or: The student does not understand the term 'Frequency'.

## Uses incorrect values to calculate the mean

For example: The student finds the total of the frequencies rather than the total number of goals.

Or: The student divides by six rather than the total frequency.

Or: The student adds the scores $(1+2+3+4+5+6)$ and divides this total by six.

Confuses the position of the median with the value for the median

For example: The student adds one to the total frequency and divides by two to give a median of 8.5 (Q1b).

Or: The student just halves the frequency (Q1b).
Or: The student assumes the median is 3.5 , half way between 1 and 6 .
Or: The student writes two values for the median, 3 and 4.

| Presents the range as two figures, the highest and the lowest scores | - What calculation is needed to obtain the range? |
| :---: | :---: |
| Calculates the range in frequencies rather than the range of goals scored | - What was the highest number of goals scored? <br> - What was the lowest number of goals scored? |
| Reads off the frequency of the tallest bar as the mode, rather than the score <br> For example: The student gives the mode as 5 (Q1b). | - Which score was the most popular? How can you tell? |
| Draws a bar chart that satisfies none or some of the criteria given in the table <br> For example: The student draws a bar chart with a mode of 4 but the other values in the table are not satisfied (Q2). | - Check that your bar chart works for all the values in the table. What is the mean/median/mode/range? <br> - Can you use the bar chart to draw a frequency table? |
| Completes the task <br> The student needs an extension task. | - Can you produce a different bar chart (to Q2) that describes the same data measures? What is the same and what is different? |

## SUGGESTED LESSON OUTLINE

## Whole-class introduction (20 minutes)

Give each student a mini-whiteboard, a pen, and an eraser.
Display Slide P-1 of the projector resource:

## Computer Games: Ratings



Imagine rating a popular computer game.
You can give the game a score of between 1 and 6.
Many students may be aware of rating systems used on popular websites. Ask students to name a computer game that most people know. If more than one computer game is suggested, then you may want to ask the class to vote on which one they want to rate.

Once the computer game has been agreed upon, ask students to rate the game by writing a score between 1 and 6 on their mini-whiteboards (if you prefer, you could use pieces of paper or card rather than whiteboards.)

How would you rate the game on a scale of 1 to 6 where $1=$ poor and $6=$ great?
On your whiteboard/paper show me your score for the game. It must be a whole number e.g. $2^{1 / 2} 2$ is not allowed.

The results of the student survey will be used to produce a bar chart from which the process of using the bar chart to find the mean, median, mode, and range will be discussed.

Before you do this, question students on efficient ways of recording the data collected in the class. The focus here is on an efficient method for collecting the scores rather than different ways of displaying the data:

You have each got a score for the game.
How can we record the scores for the class on the board?
Students may suggest writing a list of the responses or creating a tally. Discuss the benefits of using a list or tally when the data is not being collected simultaneously e.g. surveying makes of cars driving past a certain point. Emphasize the difference between this kind of data collection and the data that has just been collected by the class, whilst highlighting the importance of using an efficient method.

If students have not already suggested it, introduce the idea of a frequency table and check that students understand the term 'Frequency':

In math, what does the word 'Frequency' mean?
In this case, can you think of an equivalent phrase?
Why do we use 'Frequency' instead of (the equivalent phrase)?
[Frequency is a general term that can be used when working with data. It is usually an abbreviation of a longer, more specific phrase.]

Display Slide P-2 of the projector resource:

## Bar Chart from a Frequency Table

| Score | 1 | 2 | 3 | 4 | 5 | 6 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Frequency |  |  |  |  |  |  |



| Mean score |  |
| :---: | :--- |
| Median score |  |
| Mode score |  |
| Range of scores |  |

Use the scores on the students' whiteboards to complete the frequency table, then ask a volunteer to come out and complete the bars on the bar chart.

We are now going to find the mean, median, mode, and range of scores for the game.
Check that students understand the meaning of the terms 'mean', 'median', 'mode', and 'range'. Ask students to come out and demonstrate the four calculations using the information collected and notice whether they choose to use the frequency table or the bar chart. Emphasize using the bar chart directly as this is what the students will be doing when completing the collaborative task:

In this lesson you will be matching information displayed in a bar chart with values for the mean, median, mode and range.
Without writing anything down, how can we calculate the median, mode, and range from the bar chart?
Students may struggle to find the median directly from the bar chart and may prefer to write the data points in order and cross out until the 'middle' number is reached. If this is the case, spend some time exploring different strategies for finding the 'median' directly from the bar chart. Depending on the data collected, it may also be appropriate to discuss the method of finding the median score where there is no 'middle number'.

Hold a discussion on calculating the mean:
How can we calculate the mean score?
It is likely that students will need to write some values down when finding the mean. Spend some time discussing possible ways of calculating the mean score using the bar chart, without writing out a list of the raw scores, for example by multiplying frequencies by scores then summing.

## Collaborative small-group work: Matching Cards ( $\mathbf{3 5}$ minutes)

Organize the class into groups of two or three students and give each group Card Set: Bar Charts and Card Set: Statistics Tables (already cut-up), a large sheet of paper, and a glue stick.

Explain how students are to work collaboratively:
Take turns to match a bar chart with a statistics table. Place the cards side by side on your desk, rather than on top of one another, so that everyone can see them.
Each time you match a pair of cards explain your thinking clearly and carefully. You may want to use your mini-whiteboards for any calculations and/or when explaining to each other what you have done.
Partners should either agree with the explanation, or challenge it if it is unclear or incomplete.

Once agreed, stick the cards onto the poster paper writing any relevant calculations and explanations next to the cards.
You will notice that some of the statistics tables have gaps in them and one of the bar charts cards is blank. Try to work out what these blanks should be and complete the cards before finalizing your matches.
Slide P-3 of the projector resource summarizes these instructions.
While students are working in small groups you have two tasks: to note different student approaches to the task and to support students working as a group.

## Note different student approaches

Listen and watch students carefully. In particular, notice how students make a start on the task, where they get stuck, and how they overcome any difficulties.

Do students start with a bar chart and calculate the mean, median, mode and range and then see if there is a statistics table that matches? If so, which average do they calculate first? Do they check to see if there are any other statistics tables that might also work? Do they sort the statistics tables? If so, how? Do they use a process of elimination? If so, what measure do they use to do this? How do they use the statistics table to complete the blank bar chart card? When finding the range from a graph with a frequency of zero for either a score of 1 or 6 , do they still include these scores for the maximum or minimum values?

## Support students working as a group

As students work on the task support them in working together. Encourage them to take turns and if you notice that only one partner is matching cards, ask other students in the group to explain the match.

Carl matched these two cards. Jess, why does Carl think these two cards match?
If students in the group take different approaches when matching cards, encourage them to clearly explain the basis for a choice. Try to avoid giving students the information they need to match pairs. Instead, encourage students to interpret the cards by careful questioning, for example:

Which two bar chart cards show a sample size of 12? How do you know?
Can we tell how big the sample size is from the statistics table? Why / Why not?
Is there more than one way of completing the blank bar chart?
What is the sample size for the bar chart you have drawn?
Can you draw a bar chart with a different sample size that still satisfies the values in the statistics table?

Some groups may ignore the blanks on the cards. Check that they are filling these in as they complete their matches. It is not essential that students complete all of the matches but rather that they are able to develop effective strategies for matching the cards and can justify their matches.

You may want to draw on the questions in the Common issues table to support your own questioning.

## Whole-class discussion: sharing strategies

During the collaborative small-group work you may want to hold a brief class discussion about the strategies being used within the class. This may help students who are struggling to get started.

Jane what have you done so far? Can you explain your reasons for your chosen strategy? Has anyone else used a different strategy?

Rather than promoting a particular strategy, focus the discussion on exploring different possible methods of working:

Do you need to calculate all four statistics in turn for each bar chart? If not, why not?
Which statistic is easiest to find? How might you use this to complete the task?
What similarities or differences could you look for in the cards? How could you use what you notice?
If students are still struggling with the task, suggest that they focus on the cards with all four statistics values completed (S1, S4, S5, and S7) and see if they can find bar charts to match these four cards. Alternatively they may find it helpful to group the cards in some way e.g. all cards with a range of 3 .

## Extending the lesson over two days

If you are taking two days to complete the unit then you may want to end the first lesson here, ensuring that students have glued their matched cards onto their poster. Then, at the start of the second day, give students time to familiarize themselves with their own work before comparing posters with another group.

## Sharing posters ( 20 minutes)

Once students have finished their posters, ask them to share their work by visiting another group. This gives the students the opportunity to engage at a deeper level with the mathematics and encourages a closer analysis of the work than may be possible by students presenting their posters to the whole class. It may be helpful for students to jot down the pairs of cards matched on their mini-whiteboards, for example, B2 and S6 etc. before they visit another group.

Now, one person from each group get up and visit a different group and look carefully at their matched cards.
Check the cards and point out any cards you think are incorrect.
You must give a reason why you think the card is incorrectly matched or completed, but do not make changes to the card.
Once students have checked another group's cards, they may need to review their own cards, taking into account comments from their peers. They can make any necessary changes by drawing arrows to where a particular bar chart or statistics table should have been placed, writing a justification for the change on the poster next to the arrow.

Slide P-4 of the projector resource summarizes these instructions:

## Sharing Posters

1. One person from each group visit a different group and look carefully at their matched cards.
2. Check the cards and point out any cards you think are incorrect. You must give a reason why you think the card is incorrectly matched or completed, but do not make changes to the card.
3. Return to your original group, review your own matches and make any necessary changes using arrows to show if card needs to move.

The finished poster may look something like this:


## Whole-class discussion (20 minutes)

It is likely that some groups will not have matched all of the cards, but the aim of this discussion is not to check answers but to explore the different strategies used by students when matching/completing the cards, as well as identifying areas in which students struggled.

First select a pair of cards that most groups correctly matched. This approach may encourage good explanations. Then select one or two cards that most groups found difficult to match.

Once one group has justified their choice for a particular match, ask other students to contribute ideas for alternative strategies and their views on which reasoning method was easier to follow. The intention is that you focus on getting students to understand and share their reasoning.

Use your knowledge of the students' individual and group work to call on a wide range of students for contributions.

Which cards were the easiest to match? Why was this?
Which cards were difficult to match? Why was this?
When matching the cards, did you always start with the bar chart/statistics table? Why was this? Did anyone use a different strategy?
You may again want to draw on the questions in the Common issues table to support your own questioning.

Follow-up lesson: reviewing the assessment task ( 15 minutes)
Give each student a copy of the assessment task Boy Bands and their original scripts from the assessment task Penalty Shoot-Out. If you have not added questions to individual pieces of work, then write your list of questions on the board. Students select from this list only those questions they think are appropriate to their own work.

Read through your papers from Penalty Shoot-Out and the questions [on the board/written on your script.] Answer these questions and revise your response.
Now look at the new task sheet, Boy Bands. Can you use what you have learned to answer these questions?
If students struggled with the original assessment task, you may feel it more appropriate for them to revisit Penalty Shoot-Out rather than attempt Boy Bands. If this is the case give them another copy of the original assessment task instead.

## SOLUTIONS

## Assessment task: Penalty Shoot-Out

1a. There are 16 people involved in the shoot-out. This is the sum of all the frequencies on the bar chart.
b.

| Mean score | 3.5 |
| :---: | :---: |
| Median score | 3 |
| Mode score | 6 |
| Range of scores | 5 |

2. A possible solution is:


## Assessment task: Boy Bands

1a. There are 12 people participating in the quiz.
b.

| Mean score | 3 |
| :---: | :---: |
| Median score | 2.5 |
| Mode score | 5 |
| Range of scores | 4 |

2. A possible solution is:


Collaborative activity: Card Matching
Missing values to be completed by students are in bold.




## Penalty Shoot-Out

1. The bar chart represents the outcome of a penalty shoot-out competition.

Each person in the competition was allowed six shots at the goal.
The graph shows, for example, that four people only scored one goal with their six shots.


a. How many people were involved in the shoot-out? Show how you obtain your answer.
$\qquad$
$\qquad$
b. Complete the table with values for the Mean, Median, Mode, and Range of scores.

Explain how you calculate each answer.

2. There is another penalty shoot-out.

Use the table of results to draw a possible bar chart of the scores:

| Mean score | 3 |
| :---: | :---: |
| Median score | 3.5 |
| Mode score | 4 |
| Range of scores | 4 |



Show all your work.
$\qquad$
$\qquad$
$\qquad$
$\qquad$
$\qquad$
$\qquad$
$\qquad$

Card Set: Bar Charts


## Card Set: Bar Charts (continued)



## Card Set: Statistics Tables

| S1 |  |  | S2 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | Mean score | 3 |  | Mean score | 3 |
|  | Median score | 4 |  | Median score | 3 |
|  | Mode score | 4 |  | Mode score | 3 |
|  | Range of scores | 3 |  | Range of scores |  |
| S3 |  |  | S4 |  |  |
|  | Mean score | 3 |  | Mean score | 4 |
|  | Median score | 2 |  | Median score | 4 |
|  | Mode score |  |  | Mode score | 4 |
|  | Range of scores | 5 |  | Range of scores | 4 |
| S5 |  |  | S6 |  |  |
|  | Mean score | 3 |  | Mean score |  |
|  | Median score | 3 |  | Median score | 3 |
|  | Mode score | 4 |  | Mode score | 3 |
|  | Range of scores | 3 |  | Range of scores | 4 |

## Card Set: Statistics Tables (continued)



## Boy Bands

1. The bar chart represents the scores from a quiz.

Children were asked to name six boy bands in 30 seconds.
Each score represents the number of correctly named bands.


a. How many children were involved in the quiz? Show how you obtain your answer.
b. Complete the table with values for the Mean, Median, Mode, and Range of scores. Explain how you calculate each answer.

| Mean score |  |  |
| :---: | :---: | :---: |
|  |  |  |
| Median score |  |  |
|  |  |  |
| Mode score |  |  |
|  |  |  |
| Range of scores |  |  |
|  |  |  |

2. The results of another quiz question is shown in the table below. Draw a possible bar chart of the scores:

| Mean score | 4 |
| :---: | :---: |
| Median score | 3.5 |
| Mode score | 3 |
| Range of scores | 4 |



Show all your work.
$\qquad$
$\qquad$
$\qquad$
$\qquad$
$\qquad$
$\qquad$
$\qquad$

## Computer Games: Ratings



Imagine rating a popular computer game.
You can give the game a score of between 1 and 6.

## Bar Chart from a Frequency Table

| Score | 1 | 2 | 3 | 4 | 5 | 6 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Frequency |  |  |  |  |  |  |



| Mean score |  |
| :---: | :--- |
| Median score |  |
| Mode score |  |
| Range of scores |  |

## Matching Cards

1. Each time you match a pair of cards, explain your thinking clearly and carefully.
2. Partners should either agree with the explanation or challenge it if it is unclear or incomplete.
3. Once agreed stick the cards onto the poster and write a justification next to the cards.
4. Some of the statistics tables have gaps in them and one of the bar charts is blank. You will need to complete these cards.

## Sharing Posters

1. One person from each group visit a different group and look carefully at their matched cards.
2. Check the cards and point out any cards you think are incorrect. You must give a reason why you think the card is incorrectly matched or completed, but do not make changes to the card.
3. Return to your original group, review your own matches and make any necessary changes using arrows to show if card needs to move.

Mathematics Assessment Project

## Classroom Challenges

These materials were designed and developed by the Shell Center Team at the Center for Research in Mathematical Education University of Nottingham, England:

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The full collection of Mathematics Assessment Project materials is available from http://map.mathshell.org

