PROBLEM SOLVING

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Maximizing Area: *Gold Rush*

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Maximizing Area: *Gold Rush*

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Interpret a situation and represent the variables mathematically.
- Select appropriate mathematical methods to use and communicate their reasoning clearly.
- Explore the effects on a rectangle's area of systematically varying the dimensions whilst keeping the perimeter constant. Interpret and evaluate the data generated, identifying the optimum case.

COMMON CORE STATE STANDARDS

This lesson relates to **all** the *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 1, 3, 4, 5, and 7:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

7.G: Draw construct, and describe geometrical figures and describe the relationships between them.
Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

7.EE: Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

INTRODUCTION

- Before the lesson, students attempt the *Gold Rush* task individually. You then look at their responses and formulate questions for students to think about as they review their work.
- At the start of the lesson, students use the questions posed to think of ways to improve their work. Next, they work collaboratively to produce a better solution than they did individually.
- In a whole-class discussion students compare and evaluate the different methods they used.
- Working in the same small groups, students analyze some sample responses, then, review as a class the methods they have seen. In a follow-up lesson students reflect on their work.

MATERIALS REQUIRED

- Each individual student will need a copy of *Gold Rush*, some plain paper, a mini-whiteboard, pen, and eraser, and a copy of the review questionnaire *How Did You Work?*
- Each group of students will need a sheet of poster paper and the *Sample Responses to Discuss*.
- Provide calculators, rulers, and squared paper for students on request. String cut into equal lengths or popsicle sticks may also be helpful in representing the situation.

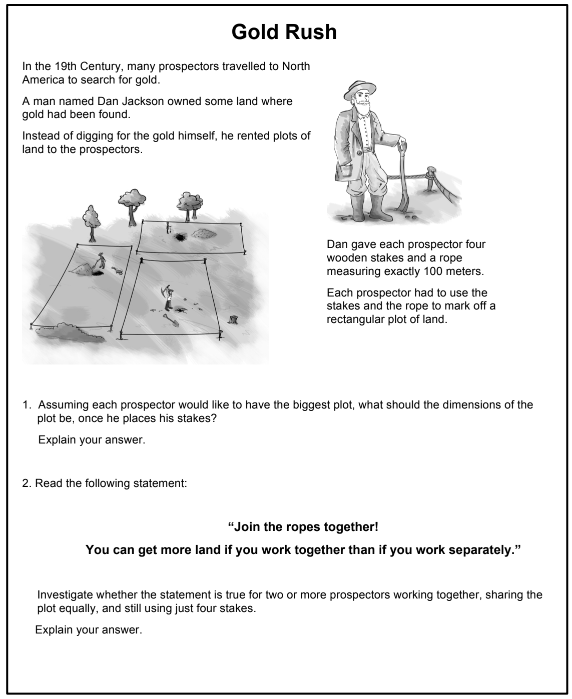
TIME NEEDED

20 minutes before the lesson, a 120-minute lesson (or two 60-minute lessons), and 10 minutes in a follow-up lesson. Timings are approximate. Exact timings will depend on the needs of your class.

BEFORE THE LESSON

Assessment task: *Gold Rush* (20 minutes)

Give the students this task to do, either in class or for homework, a day or more before the lesson. This will give you an opportunity to assess their work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.



Gold Rush

In the 19th Century, many prospectors travelled to North America to search for gold.

A man named Dan Jackson owned some land where gold had been found.

Instead of digging for the gold himself, he rented plots of land to the prospectors.

Dan gave each prospector four wooden stakes and a rope measuring exactly 100 meters.

Each prospector had to use the stakes and the rope to mark off a rectangular plot of land.

1. Assuming each prospector would like to have the biggest plot, what should the dimensions of the plot be, once he places his stakes?

   Explain your answer.

2. Read the following statement:

   "Join the ropes together!

   You can get more land if you work together than if you work separately."

   Investigate whether the statement is true for two or more prospectors working together, sharing the plot equally, and still using just four stakes.

   Explain your answer.

Give each student a copy of the *Gold Rush* task and some plain paper to work on. Introduce the task briefly, helping the class to understand the problem and its context.

When these prospectors dig for gold, they first mark out their plot of land and then dig inside the borders of the plot.

One of the ways of marking out a plot is to put stakes in the ground at the corners of the plot and then wrap a piece of rope around to mark out the plot's perimeter.

In this task we will be finding out how prospectors can maximize the area in which they dig for gold, when they have a fixed perimeter.

Now explain what you are asking students to do.

Read through the questions carefully and try to answer them as well as you can. Show all your work so that I can understand your reasoning and try to show your findings in an organized way. Don't worry if you can't do everything. There will be a lesson on this material [tomorrow] that will help you improve your work.

It is important that, as far as possible, students are allowed to answer the questions without assistance. If students are struggling to get started then ask questions that help them understand what is required, but make sure you do not do the task for them.

Students who sit together often produce similar responses and then, when they come to compare their work, they have little to discuss. For this reason we suggest that, when students do the task individually, you ask them to move to different seats. At the beginning of the formative assessment lesson allow them to return to their usual seats. Experience has shown that this produces more profitable discussions. When all students have made a reasonable attempt at the task, tell them that they will have time to revisit and revise their solutions later.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their problem solving strategies.

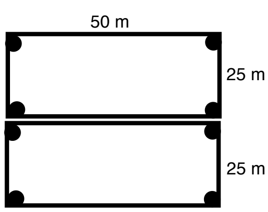
We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given in the *Common issues* table below. We suggest that you make a list of your own questions, based on your students' work, using the ideas on the following pages. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight questions for individual students.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these questions on the board when you return the work to the students at the beginning of the lesson.

Common issues:	**Suggested questions and prompts:**
Does not understand the concept of area and/or perimeter or does not know how to find the area and perimeter of a rectangle	• What does the length of the rope given to a prospector measure? • How could you measure the amount of land enclosed by the rope? • How do you find the area of a rectangle? • How do you find the perimeter of a rectangle?
Calculates the total amount of land, but not the amount of land for each prospector (Q2)	• You've worked out the total area of land for both/all the prospectors; how much land will each prospector get?
Emphasizes only the human impact of sharing the land (Q2) For example: The student states that when two people share they can help each other out. Or: The student states that when sharing the land people are more likely to steal from each other.	• Now investigate if combining ropes affects how much land each prospector gets.
Does not investigate any or very few rectangles For example: The student draws just one rectangle and calculates its area (Q1).	• Now investigate the area of several different rectangles with the same perimeter, but different dimensions.
Works unsystematically	• How can you now organize your work? • How do you know for sure your answer is the best option?
Presents work poorly For example: The student presents the work as a series of unexplained numbers and/or calculations.	• Would someone unfamiliar with this work understand your method?
Only investigates two prospectors sharing land	• Suppose 3/4/5 prospectors share land. What area of land would each prospector get?

Common issues:	**Suggested questions and prompts:**
Makes assumptions without justifying them For example: The student correctly assumes, having tried a few different rectangles, that a square gives the maximum plot area, but does not explain why this is the case (Q1). Or: The student assumes that, when two prospectors join their ropes together the area for each prospector will increase/remain the same, but does not support this with math (Q2). Or: The student correctly concludes that each time you add another person each person gets an extra 625 m^2 of land, but does not support this with math (Q2).	• Can you see a pattern in the rectangles that shows why the square is best? • Can you put your results into some kind of order? • Can you use math to explain why joining the ropes together gives each prospector a bigger area/the same area? • If you join two ropes together, how long will the rope be? • Show me an example of a plot of land using two ropes. What area of land will each prospector get? How do you know? • Suppose 3/4/5/*n* prospectors share land. What area of land would each prospector get? How do you know?
Omits or uses incorrect units For example: The student describes both length and area in meters.	• What units do we use for area? • What units do we use for perimeter?
Uses rope to divide multiple areas when investigating two or more prospectors For example: The student draws two 25 by 50 meter rectangles next to each other (Q2). 	• Can you show where the **four** stakes will go to mark off this plot? • What is the benefit of combining the ropes in this way? • Where would the two ropes be joined together?
Correctly answers all the questions Student needs an extension task.	• If the prospectors were able to use all the stakes available to them (i.e. 2 prospectors have a combined total of 8 stakes, 3 prospectors have a combined total of 12 stakes etc.) what other shapes could be investigated? • What effect would this have on the area of land available to each prospector?

SUGGESTED LESSON OUTLINE

This lesson assumes students understand the concept of area and perimeter and how to find the area and perimeter of a rectangle.

Introduction: *Gold Rush* task (10 minutes)

Give each student a mini-whiteboard, a pen, and an eraser. Begin the lesson by briefly reintroducing the problem. You may want to show the class Slide P-1 of the projector resource.

Recall the Gold Rush problem. What was the task about?

Today we are going to work together to try to improve your initial attempts at this task.

First, I have had a look at your work and have some questions I would like you to think about.

On your own, carefully read through the questions I have written. I would like you to use the questions to help you to think about ways of improving your own work.

Use your mini-whiteboards to make a note of anything you think will help improve your work.

Return your students' work on *Gold Rush*.

If you have not added questions to individual pieces of work or highlighted questions on a printed list of questions then write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

Collaborative activity: producing a joint solution (35 minutes)

Organize students into groups of two or three. Give each group a large sheet of paper for making a poster. You may also want to give out some lengths of string.

Deciding on a Method

I want you to share your method with your group.

Take turns to explain your method and how you think your work could be improved.

Listen carefully to each other. Ask questions if you don't understand.

Once everyone in the group has explained their method, I want you to come up with a joint method that is better than your separate ideas.

Slide P-2 of the projector resource summarizes these instructions.

To confirm students know what they have to do, ask a couple of students to explain, in succession, the different steps of the activity.

Once students have evaluated the relative merits of each approach and decided on a joint method, ask them to write a brief outline of their chosen method on one side of their sheet of poster paper, giving clear reasons for their choice of method.

Implementing the Method

Together in your group, produce a poster showing your joint solution. State on your poster any assumptions you have made.

While students work in small groups you have two tasks: to note their different approaches to the task and to support their reasoning.

Note different student approaches to the task

Listen and watch students carefully. Note different approaches to the task and what assumptions students make. Do students understand how to find the area and perimeter of a rectangle? Do students draw an appropriate number of rectangles and collect the data in an organized way, for example, a table? Do they give reasoned explanations for their findings and explain their work clearly? Do they work systematically? Do they make any attempts to generalize?

Support student reasoning

If students are struggling to produce a joint solution to the task, try not to make suggestions that move students towards a particular approach. Instead, ask questions that help them to clarify their thinking. If the whole-class is struggling on the same issue, write relevant questions on the board and hold an interim discussion.

To further help students struggling with the task:

- Use the questions in the *Common issues* table to support your own questioning.
- Ask students who performed well on the assessment to help struggling students.
- Hand out one or two pieces of sample work.

Whole-class discussion (15 minutes)

When students have had sufficient time to work on their posters, hold a whole-class discussion to review how they have worked. Have students solved the problem using a variety of methods? Or have you noticed some interesting ways of working or some incorrect methods, if so, you may want to focus the discussion on these. Equally, if you have noticed different groups use similar strategies but make different assumptions, you may want to compare solutions.

You may want to draw on the questions in the *Common issues* table to support your own questioning.

Extending the lesson over two days

If you are taking two days to complete the unit then you may want to end the first lesson here. At the start of the second day, briefly remind students of the problem before moving on to the collaborative analysis of sample responses.

Collaborative analysis of *Sample Responses to Discuss* (40 minutes)

Distribute copies of the *Sample Responses to Discuss* to each group of students. This task gives students an opportunity to evaluate a variety of possible approaches to the task, without providing a complete solution strategy.

There may not be time and it is not essential for all groups to look at all four sample responses. If this is the case, be selective about what you hand out. For example, groups that have successfully completed the task using one method will benefit from looking at different approaches. Other groups that have struggled with a particular approach may benefit from seeing another student's work that uses the same strategy.

Here are some different responses to the problem.

Imagine you are the teacher and you have to review the work.

Go through each piece of work and answer the questions below the work.

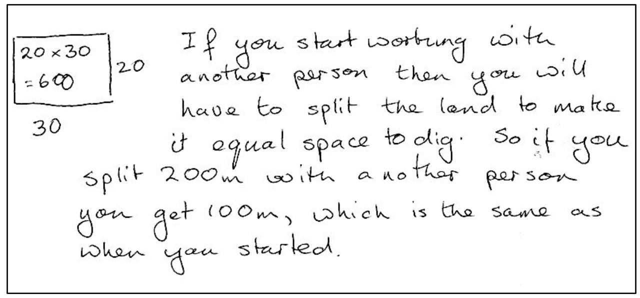
Slide P-3 of the projector resource describes how students are to work together.

Encourage students to focus on evaluating the math contained in the student work, not whether the student has neat writing etc.

During the small group work, support the students as in the first collaborative activity. Also, check to see which of the explanations students find more difficult to understand. Note similarities and differences between the sample approaches and those the students took in the collaborative group work.

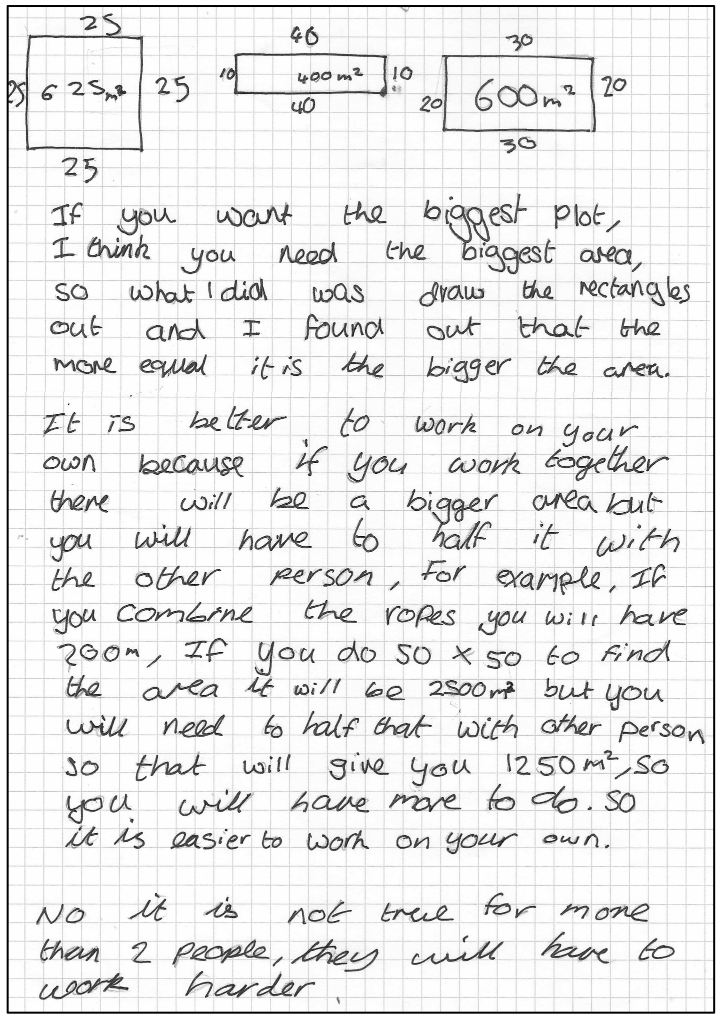
Ann draws a rectangle with the correct perimeter and finds its area. However, she does not try any other rectangles to see if their areas are different.

When considering two prospectors sharing, Ann confuses perimeter and area.



Jake has drawn three rectangles, including the square. He has correctly concluded that "the more equal it is, the bigger the area" but does not state that a square is biggest. In the later sections, he seems more concerned about reducing the prospectors' workload than giving them the best chance of finding gold.

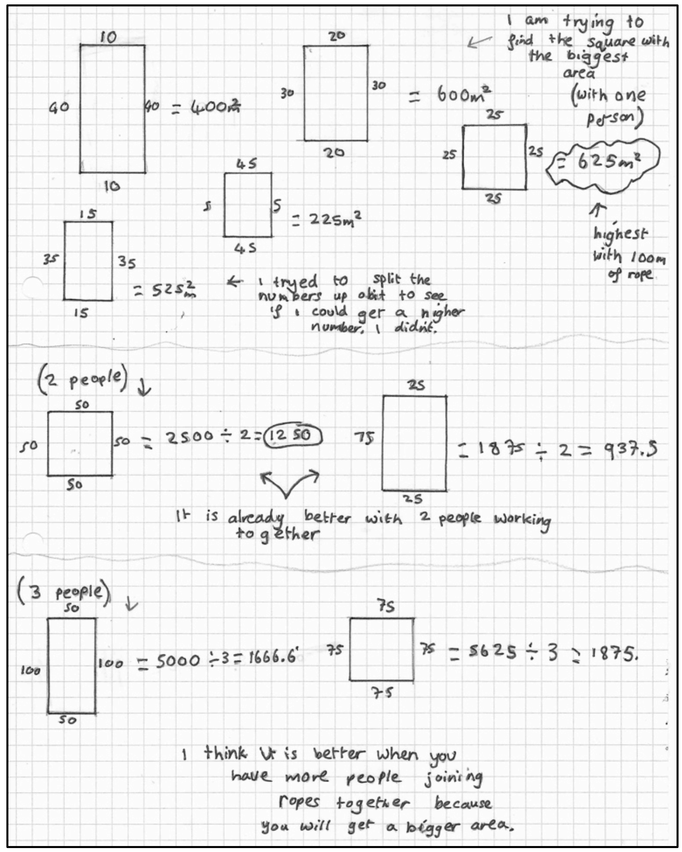
In what way has Jake misinterpreted the question?



Kodie has drawn three rectangles, including a square and concluded that the biggest area is the square. She has then tried to be sure of this, by drawing two more rectangles.

She has tried two rectangles, including a square, for two and three people and has remembered to work out how much land each would have.

Her final conclusion is correct. Kodie has attempted to generalize, but this has not been formalized.

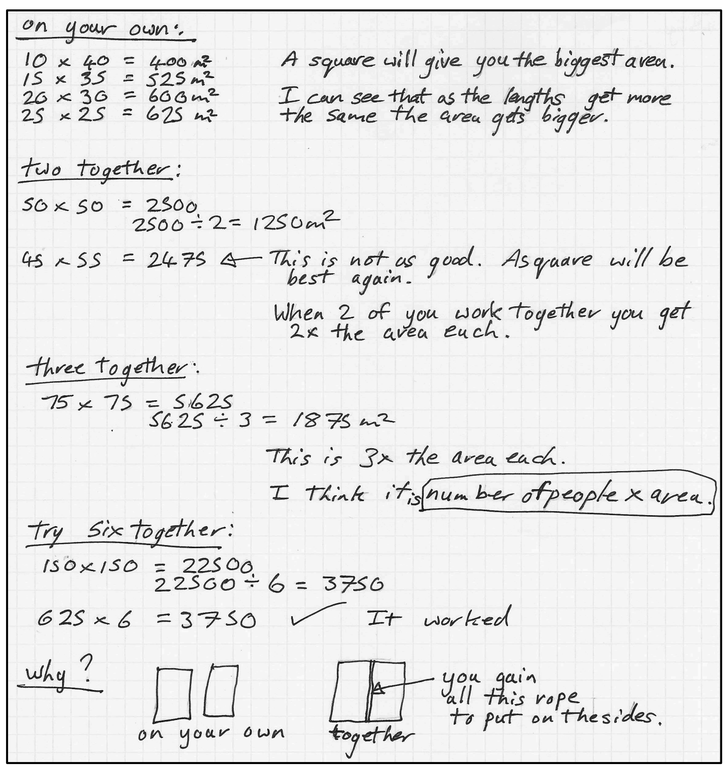


Mark has used rectangles with the difference in side lengths getting progressively less. He has noted that the areas increase and concluded that the square gives the biggest area.

He finds the areas for two and three ropes and sees a pattern in the increasing areas, which he checks using six ropes. He considers why the areas increase and gives a reason.

How could Mark prove the rule he has found using algebra?

[n prospectors can make a square of perimeter $100n$. This gives a side length of $25n$ and a total area of $625n^2$. Sharing this total area between the n prospectors means that each gets $625n$, or n times what the prospector would have had before sharing.]



Whole-class discussion: comparing different solution methods (20 minutes)

Organize a whole-class discussion to consider the different approaches used in the sample work. Focus the discussion on parts of the task students found difficult. Ask the students to compare the different methods.

Which method did you like best? Why?

Which method did you find most difficult to understand? Why?

How could the student improve his/her answer?

Did anyone come up with a method different from these?

Try to focus the discussion on any common misconceptions you noticed in the first collaborative activity. Again you may want to draw on the questions in the *Common issues* table to support your own questioning.

You could use Slides P-4 –P-7 of the projector resource to support your discussion.

Depending on your class, you may want to also discuss whether the students make any incorrect assumptions, whether they work systematically, and whether they attempt to generalize.

Follow-up lesson: review of work (10 minutes)

Give a copy of the questionnaire *How Did You Work?* to each student and ask them to complete it.

The questionnaire may help students to monitor and review their progress during and at the end of an activity.

If you have time you may also want to ask your students to read through their original method and using what they have learned, attempt the task again.

Some teachers give this task as homework.

SOLUTIONS

1. For a prospector to have the biggest plot, the rope needs to be laid out in a square of side 25 m, with one of the four stakes at each corner. This will give the prospector an area of 25 m × 25 m = 625 m^2.

   The following table may be helpful in checking students' area calculations. It is likely that students will have calculated just a few of these areas.

Length of Rope (m)	Length (m)	Width (m)	Area (m^2)
100	1	49	49
100	2	48	96
100	3	47	141
100	4	46	184
100	5	45	225
100	6	44	264
100	7	43	301
100	8	42	336
100	9	41	369
100	10	40	400
100	11	39	429
100	12	38	456
100	13	37	481
100	14	36	504
100	15	35	525
100	16	34	544
100	17	33	561
100	18	32	576
100	19	31	589
100	20	30	600
100	21	29	609
100	22	28	616
100	23	27	621
100	24	26	624
100	25	25	625

2. When two prospectors tie their ropes together the new rope length is 200 m. A square plot again results in the largest area:

   50 m × 50 m = 2500 m^2.

   Dividing this area between the two prospectors gives each one an area of 1250 m^2, which is twice the area they would have if they were working on their own. Therefore, the proposition is true for two people!

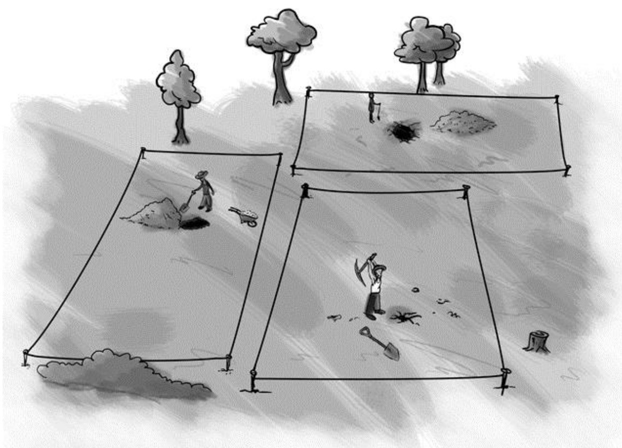
   In general, the area received by each prospector when n prospectors join their ropes together is $625n$. The reasoning goes as follows: n prospectors can make a square of perimeter $100n$. This gives a side length of $25n$ and a total area of $625n^2$. Sharing this total area between the n prospectors means that each gets $625n$, or n times what the prospector would have had before sharing.

Gold Rush

In the 19th Century, many prospectors travelled to North America to search for gold.

A man named Dan Jackson owned some land where gold had been found.

Instead of digging for the gold himself, he rented plots of land to the prospectors.



Dan gave each prospector four wooden stakes and a rope measuring exactly 100 meters.

Each prospector had to use the stakes and the rope to mark off a rectangular plot of land.

1. Assuming each prospector would like to have the biggest plot, what should the dimensions of the plot be, once he places his stakes?

   Explain your answer.

2. Read the following statement:

"Join the ropes together!

You can get more land if you work together than if you work separately."

Investigate whether the statement is true for two or more prospectors working together, sharing the plot equally, and still using just four stakes.

Explain your answer.

Sample Responses to Discuss: Ann

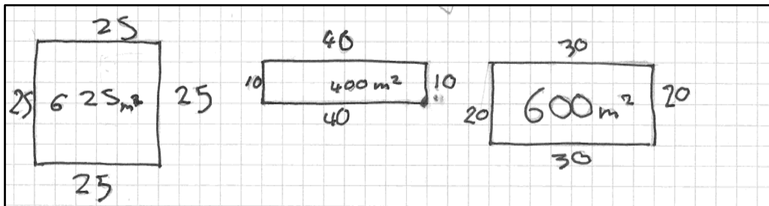
20 × 30 = 600 | 20
30

If you start working with another person then you will have to split the land to make it equal space to dig. So if you split 200m with another person you get 100m, which is the same as when you started.

What Math did Ann do well?

What mistakes has Ann made?

Sample Responses to Discuss: Jake



If you want the biggest plot, I think you need the biggest area, so what I did was draw the rectangles out and I found out that the more equal it is the bigger the area.

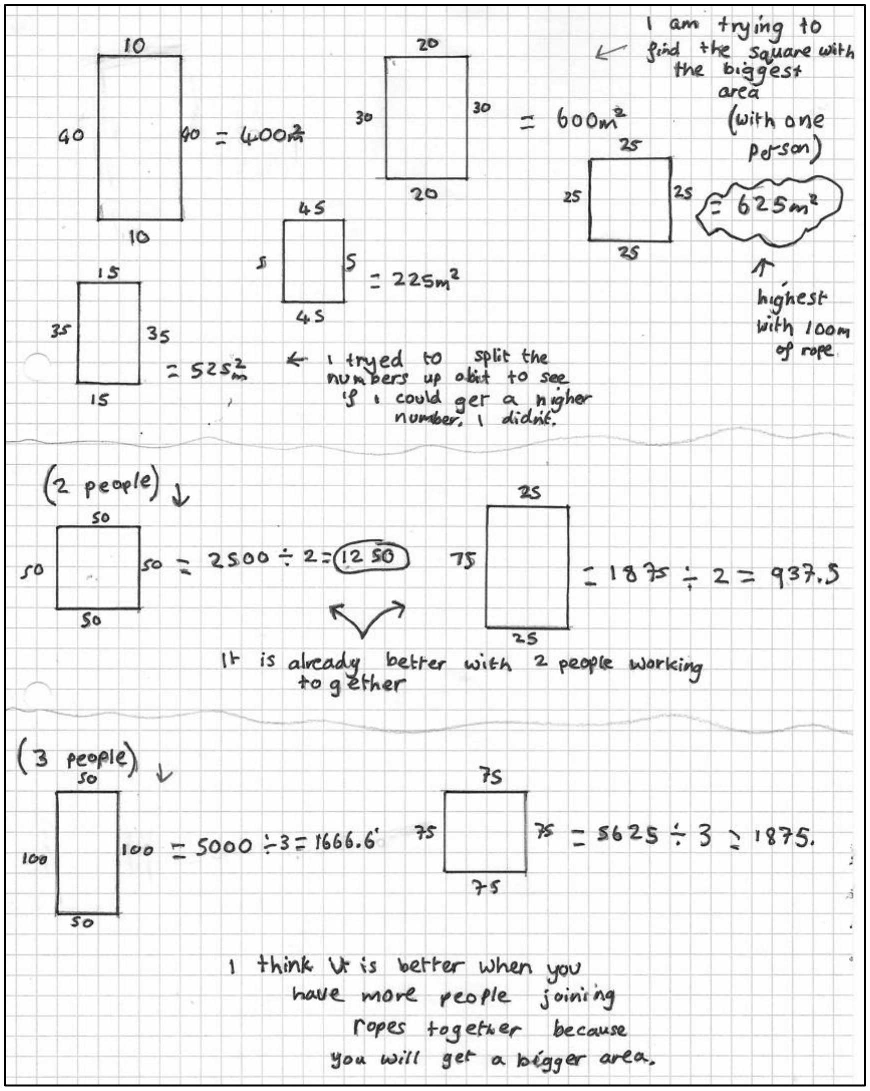
It is better to work on your own because if you work together there will be a bigger area but you will have to half it with the other person, For example, If you combine the ropes you will have 200m, If you do 50 × 50 to find the area it will be 2500m² but you will need to half that with other person so that will give you 1250m², so you will have more to do. So it is easier to work on your own.

No it is not true for more than 2 people, they will have to work harder.

What Math did Jake do well?

Has Jake made any incorrect assumptions? How else could Jake improve his work?

Sample Responses to Discuss: Kodie



What Math did Kodie do well?

How could Kodie improve the presentation of her work?

What further work could Kodie do?

Sample Responses to Discuss: Mark

on your own:

10 × 40 = 400 m²
15 × 35 = 525 m²
20 × 30 = 600 m²
25 × 25 = 625 m²

A square will give you the biggest area.

I can see that as the lengths get more the same the area gets bigger.

two together:

50 × 50 = 2500
2500 ÷ 2 = 1250 m²

45 × 55 = 2475 ← This is not as good. A square will be best again.

When 2 of you work together you get 2× the area each.

three together:

75 × 75 = 5625
5625 ÷ 3 = 1875 m²

This is 3× the area each.

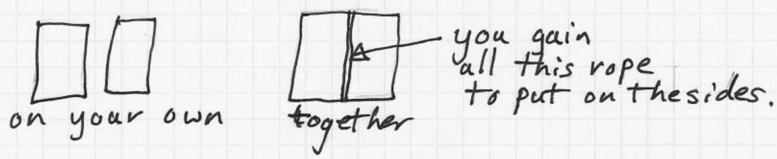
I think it is number of people × area.

try six together:

150 × 150 = 22500
22500 ÷ 6 = 3750

625 × 6 = 3750 ✓ It worked

why?



What Math did Mark do well?

How could Mark improve his work?

How Did You Work?

Tick the boxes and complete the sentences that apply to your work.

1. Our group work was better than my own individual work ☐

   Our joint solution was better because ....

   ....

2. We justified our solution ☐

   We justified our solution by ....

   ....

3. Our method is similar to one of the sample responses ☐ **OR** Our method is different from **all** of the sample responses ☐

   Our method is similar to .... *(add name of sample response)*

   I prefer **our method / the sample response** (*circle*)

   This is because ....

   ....

   ....

   Our method is different from all of the sample responses

   because ....

   ....

   ....

   ....

4. In our method we assumed that:

   ....

   ....

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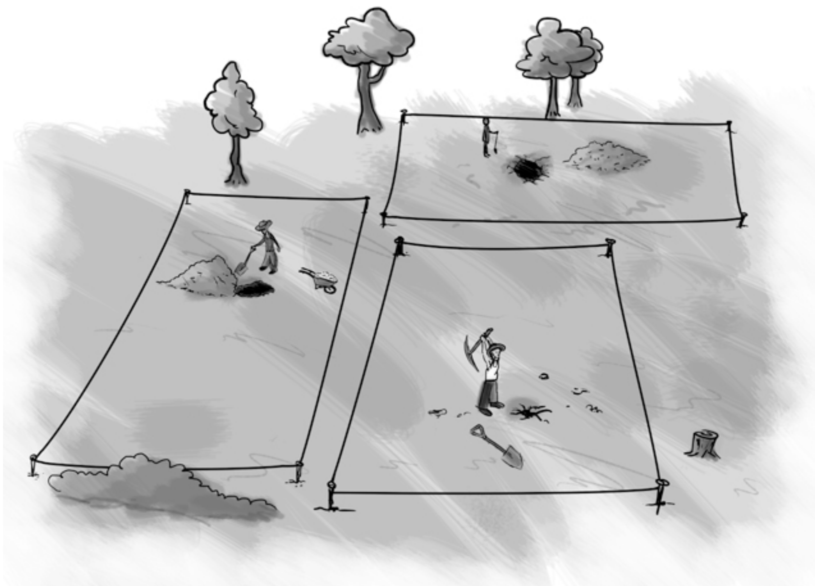


Dan gave each prospector four wooden stakes and a rope measuring exactly 100 meters.

Each prospector had to use the stakes and the rope to mark off a rectangular plot of land.

Planning a Joint Method

1. Take turns to explain your method and how you think your work could be improved.

2. Listen carefully to each other.
   - Ask questions if you don't understand.

3. Once everyone in the group has explained their method, plan a joint method that is better than each of your separate ideas.

4. Make sure that everyone in the group can explain the reasons for your chosen method.

5. Write a brief outline of your method on one side of your sheet of paper.

Evaluating Student Sample Responses

1. Imagine you are the teacher and have to assess the student work.

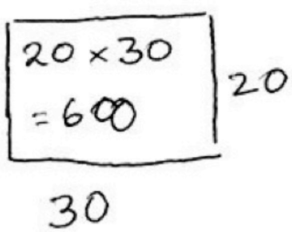
2. Take it in turns to work through a students' solution.
   - Write your answers on your mini-whiteboards.

3. Explain your answer to the rest of the group.

4. Listen carefully to explanations.
   - Ask questions if you don't understand.

5. Once everyone is satisfied with the explanations, write the answers below the student's solution.
   - Make sure the student who writes the answers is not the student who explained them.

Sample Responses to Discuss: Ann



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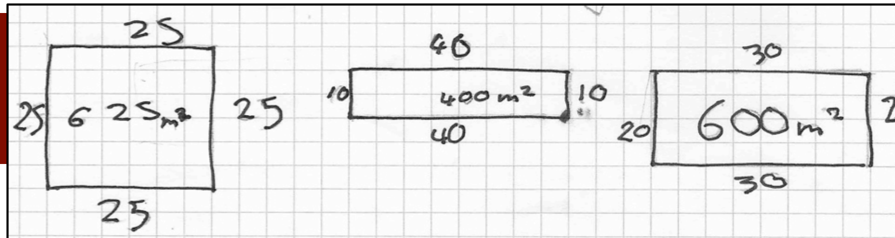
Jake's method



If you want the biggest plot, I think you need the biggest area, so what I did was draw the rectangles out and I found out that the more equal it is the bigger the area.

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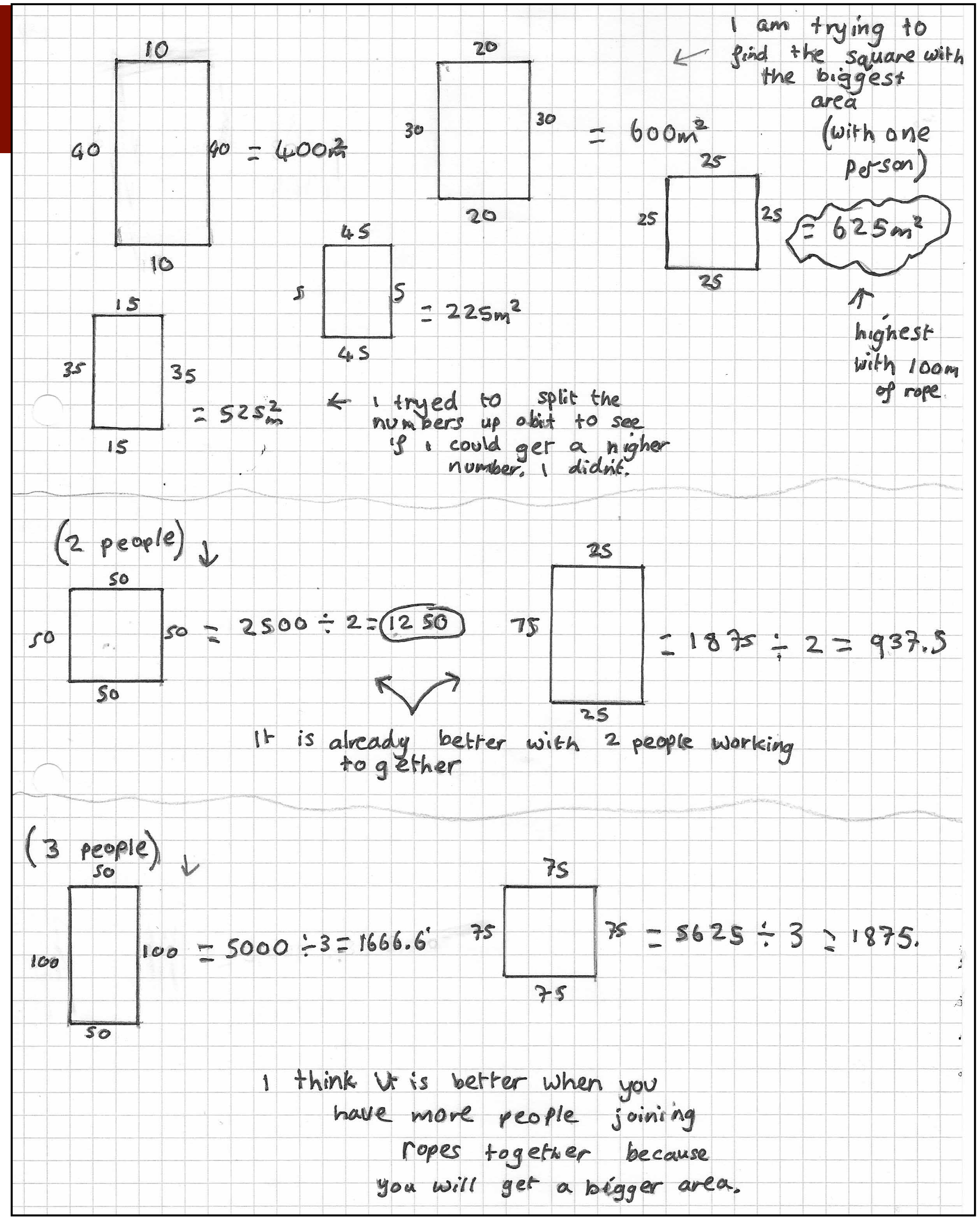
Kodie's method



Mark's method

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Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org