CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Applying Properties of Exponents

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Applying Properties of Exponents

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Recall and use the properties of exponents to generate equivalent numeric expressions.
- Identify the appropriate property to use and apply it correctly.
- Check the numerical value of an expression involving exponents without using a calculator.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards* for Mathematics:

8.EE: Work with radicals and integer exponents.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 3, 7, and 8:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

- Before the lesson, students work individually on an assessment task designed to reveal their current understanding. You then review their responses and create questions for students to consider when improving their work.
- After a whole-class introduction, students work in small groups on a collaborative discussion task, grouping cards based on numerical equivalence. Throughout their work, students justify and explain their thinking and reasoning.
- Students review their work by comparing their card groupings with their peers'.
- In a whole-class discussion, students discuss what they have learned.
- In a follow-up lesson, students revisit their initial work on the assessment task and work alone on a similar task to the introductory task.

MATERIALS REQUIRED

- Each student will need a mini-whiteboard, pen, and wipe, copies of the assessment tasks *Properties of Exponents* and *Properties of Exponents (Revisited)*.
- Each small group of students will need cut-up copies of *Card Set: Expressions* and *Card Set: Single Exponents*, a large sheet of poster paper, and a glue stick.

TIME NEEDED

10 minutes before the lesson, a 90-minute lesson (or two shorter ones), and 15 minutes in a follow-up lesson (or for homework). These timings are not exact. Exact timings will depend on the needs of your students.

BEFORE THE LESSON

Assessment task: *Properties of Exponents* (10 minutes)

Have the students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.

Give each student a copy of the assessment task *Properties of Exponents* and briefly introduce the task:

> *In the first task, you have to fill in the missing integer exponents (or powers) by writing numbers on the dotted lines*
>
> *In the second task you have to place five numbers in order of size.*
>
> *Make sure you show your working and explain your method clearly.*

It is important that, as far as possible, students are allowed to answer the questions without assistance.

Properties of Exponents

1. In each of the following questions write the missing exponents on the dotted lines. Show your reasoning in the spaces provided on the right.

a) $2 + 2 + 2 + 2 = 2^{\dots}$

b) $2 \times 2 \times 2 \times 2 = 2^{\dots}$

c) $2^{\dots} \times 2^3 = 2^6$

d) $2^3 \times 3^3 = 6^{\dots}$

e) $4^3 = 2^{\dots}$

f) $(3^{\dots})^3 = 3^6$

g) $5^6 \div 5^2 = 5^{\dots}$

h) $5^2 - 3^2 = 2^{\dots}$

i) $3^5 \div 3^{\dots} = 3^{\dots} = \frac{1}{3}$

2. Write these five numbers in order of size, from smallest to greatest:

6^0 0^6 3^2 2^3 7^{-1}

Smallest				Greatest

Show your reasoning here:

....

....

Students should not worry too much if they cannot understand or do everything because in the next lesson they will work on a similar task that should help them. Explain to students that by the end of the next lesson they should be able to answer questions such as these more confidently. This is their goal.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches.

We suggest that you do not score students' work. Research shows that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics. Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, *or*
- give each student a printed version of your list of questions, and highlight appropriate questions for each student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students, and write these on the board when you return the work to the student in the follow-up lesson.

Common issues	Suggested questions and prompts
Applies the properties $a^m \times a^n = a^{m+n}$ or $a^m \div a^n = a^{m-n}$ incorrectly The student does not appear to understand the conditions under which it is possible to **add** or **subtract** exponents. For example, the student writes: $2+2+2+2 = 2^4$ (Q1a) *or* $2^2 \times 2^3 = 2^6$ (Q1c) or $5^6 \div 5^2 = 5^3$ (Q1g) *or* $5^2 - 3^2 = 2^0$ (Q1h)	• What does the exponent tell you? • What is the difference between 5^2 and 5x2? • Write out $2^3 \times 3^3$ as a product of integers. Now write out 6^6 as a product of integers. Are these equal? • What does 5^6 mean? If I divide this by 5, what is the result? How would you write this as a power of 5? • What is the answer to $2^3 - 2^2$? How would you write this as an integer? How would you write this as a power of 2? Why should you not subtract the powers?
Applies the property $a^m \times b^m = (ab)^m$ incorrectly The student does not appear to understand the conditions under which it is possible to **multiply** the base. For example, the student writes: $2^3 \times 3^3 = 6^6$ or $2^3 \times 3^3 = 6^9$ or takes 2^3 to be 6.	• What is the value of 2^3? What is the value of 3^3? What is the value of $2^3 \times 3^3$? What is the value of 6^6?
Applies the property $(a^m)^n = a^{mn}$ incorrectly The student does not appear to understand the conditions under which it is possible to **multiply** *or* **divide** exponents. For example, the student writes: $4^3 = (2^2)^3 = 2^5$ (Q1e) *or* $(3^3)^3 = 3^6$ (Q1f)	• What does $(2^2)^4$ mean? Can you write it out as a product of integers? Is this equal to 2^6?
Misinterprets a^{-n} or fails to use negative exponents. For example, the student fails to answer (Q1i) or thinks that 7^{-1} is the smallest number (possibly -7) in (Q2).	• What is the value of 2^5? What is the value of 2^6? What is the value of $2^5 \div 2^6$? How can you write that as a power of 2? • Can you explain the difference in meaning of 7^{-1} and -7?
Confuses a^0 and 0^a and believes they are both zero. For example, the student places 6^0 and 0^6 in the same answer box (Q2).	• How does each row in this sequence lead to the next row? What does the last row tell you? $2^4 \div 2^1 = 8 = 2^3$ $2^3 \div 2^1 = 4 = 2^2$ $2^2 \div 2^1 = 2 = 2^1$ $2^1 \div 2^1 = ? = 2^?$ • How would you write 0 x 0 x 0 x 0 using exponents? What is its value?
Provides insufficient justification For example: The student correctly completes the missing exponent in Q1e giving only the justification that $4 \div 2 = 2$ so $3 \times 2 = 6$.	• Can you explain why the exponent has doubled? Does your method always work? Is 8^2 the same as 2^8? (i.e. $8 \div 2 = 4$ so $2 \times 4 = 8$?) • Can you express 4 as a power of 2? [2^2]

SUGGESTED LESSON OUTLINE

Whole-class introduction (15 minutes)

Give each student a mini-whiteboard, pen, and wipe. Check that students understand what is meant by ‘power of 2’.

What do we mean when we talk about ‘powers of 2’?
Can someone give me an example of a ‘power of 2’?

A common misconception is to think that, for example, 5^2 is a power of 2, when in fact it is a power of 5. Display Slide P-1 of the projector resource:

Powers of 2

A: $8 \times 4 = 32$

B: $16 \div 8 = 2$

C: $8 \div 16 = ½$

D: $8 \div 8 = 1$

Here we have some calculations containing numbers that can all be written as powers of 2. Working on your own, on your mini-whiteboard, write each of the calculations A, B, C and D – the whole calculation, not just the answer – using powers of 2 only.

[A: $2^3 \times 2^2 = 2^5$; B: $2^4 \div 2^3 = 2^1$; C: $2^3 \div 2^4 = 2^{-1}$; D: $2^3 \div 2^3 = 2^0$.]

Now share your work with someone sitting near to you.

Are your calculations the same?

Discuss anything that you notice.

When students have had sufficient time to discuss their work briefly, collect responses for the calculations from around the class and write them on the board.

Amy, what did you write down for calculation A/B/C/D?

Did your partner agree?

If not, what do you think the calculation should be?

How could you predict the final power of 2 from the calculation?

The product of powers property, the quotient of powers property and the property of any number raised to the power of 0 being equal to 1 may well be discussed here. Encourage students to formulate some of the properties through questioning.

Can you show me another multiplication example using powers of 2?

Show me the version using integers and also using powers of integers.
[$16 \times 4 = 64$; $2^4 \times 2^2 = 2^6$ etc.]

Chris, can you show me a more difficult example?

Jed, can you show me an example that has the answer 2^{-3}?

What methods can you use to make up these questions?

If students are very unsure at this stage, do not worry or try to address their misconceptions now. These will be tackled in the next section of the lesson.

Collaborative small-group work (30 minutes)

Ask students to work in groups of two or three. Give each group cut-up copies of *Card Set: Expressions* and *Card Set: Single Exponents*, a large sheet of poster paper, and a glue stick.

There are two card sets: E and S.

I want you to sort these cards into groups.

Each group of cards should have the same value.

Please note that some groups may contain three or four cards!
For example, there might be different expression cards with the same value.

There are also blank cards for you to make up some examples of your own.

Explain how students are to work together, using slide P-2.

Working Together

Take it in turns to:

1. Select an expression card and find **all other cards** that have the same value as the one you have chosen.
2. Explain your matching to your partner.
3. Your partner must check your matching and challenge your explanation if they disagree.
4. Once agreed, glue the cards onto the poster and record your explanation for each match.
5. Continue to take turns until you have ten groups of cards.

The purpose of this task is to encourage each student to engage with their partner's explanations and to take responsibility for their partner's understanding. Students may use their mini-whiteboards for calculations and to explain their thinking to each other, or you might want them to write out their reasoning and their developing ideas on paper. We suggest that you do not give out calculators, as this could encourage students to work out numerical values for each card. It is best to discourage this approach and to focus students instead on the structure of the calculations rather than the answers.

While students are working you have two tasks: to note different student approaches to the task and to support student problem solving.

Note different student approaches

Listen to and watch students carefully. Notice how students make a start on the task, where they get stuck, and how they overcome any difficulties. Which expressions do they choose to work with first? Do students begin by first calculating the numerical value of the expression or do they apply an appropriate exponent property? Are they writing the meanings for expressions? Are students checking their work? If so, what method(s) are they using?

In particular, notice whether students are addressing the difficulties they experienced in the assessment task. Note also any common mistakes. You may want to use the questions in the *Common issues* table to help address any misconceptions that arise.

Support student problem solving

Help students to work constructively together. Remind them to look at the slide for instructions on how to work. Check that students listen to each other and encourage them to do any necessary calculations on their mini-whiteboards.

Try not to solve students' problems or do the reasoning for them. Instead, you might ask strategic questions to suggest ways of moving forward.

What do you know about this expression?

Can you write out this expression in full, without using exponents?

What are you able to work out?

Are there any more expressions that look like this? Does the same method apply?

Some groups may not manage to group all of the cards and it is not essential that they do so. It is more important that every student understands the equivalence of each set of grouped cards.

If a group of students finishes matching a set of cards, then ask them to create some new ones that will match the same set, or to create a new set of matching cards. Some blank cards are included for this purpose.

Extending the lesson over two days

If you are taking two days to complete this lesson unit then you may want to end the first lesson here. At the start of the second day, briefly remind students of the task they have been working on before moving on students sharing their work.

Sharing work (15 minutes)

As students finish their posters, have them share and comment on each other's work by asking one student from each group to visit another group. It may be helpful for the students visiting another group to record first a list of their grouped cards on their mini-whiteboard (e.g. E4 & S4, E14, E5 & S7 etc.). Slide P-3 of the projector resource, *Sharing Work*, explains the process:

> **Sharing Work**
>
> One person in your group:
>
> - Write down your card matches on your mini-whiteboard.
> - Go to another group's desk and compare your work with theirs.
> - If there are differences in your matches, ask for an explanation.
> - If you still don't agree, explain your own thinking.
>
> The other person(s) should:
>
> - Stay at your desk, and be ready to explain the reasons for your group's decisions.

If peer review is new to your students, you might need to model briefly how to review another group's work, such as by using two groups' work as an example and having a short whole-class discussion about differences and inviting explanations.

Poster Review (10 minutes)

Once students have had a chance to share their work and discuss their groupings and reasoning with their peers, give them a few minutes to review their posters.

Now that you have discussed your work with someone else, you need to consider as a group whether to make any changes to your own work.

If you think a card is in the wrong place, draw an arrow on your poster to show which group it belongs to.

Whole-class discussion (20 minutes)

Conduct a whole-class discussion about what has been learned and explore the different methods of justification used when grouping cards. Have students used a variety of methods? Have you noticed some interesting ways of working or some incorrect methods? If so, you may want to focus the discussion on these.

You may want to start the discussion by first selecting an expression that most students grouped correctly, as this may encourage good explanations.

Which cards have you matched with this expression? Can you explain why they are equivalent?

Has anyone found different cards that are equivalent to this expression?

Does anyone have a different explanation?

Discuss a range of types of expression cards and make some variations. Then draw out generalizations:

Can someone explain why these four cards match?

E12	E10	S6	S10
$2^3 \times 2^3$	$(2^3)^2$	2^6	4^3

OK, now suppose I change E12 to $2^4 \times 2^4$.

Can you suggest three more cards that would match this?

How would I have to change E10? S6? S10?

What if we changed E12 to $2^{100} \times 2^{100}$*?* $2^n \times 2^n$*?* $[2^n \times 2^n = (2^n)^2 = 2^{2n} = (2^2)^n]$

Once one group has justified their choice for a particular expression, ask other students to contribute ideas of alternative approaches and their views on which reasoning was easier to follow. You might want to develop a set of general laws with the class as the discussion proceeds (e.g. $a^m \times a^n = a^{m+n}$ $a^m \times b^m = (a \times b)^m$).

Students may know these mechanically as the 'rules of exponents'. You could now see whether they are able to justify these rules:

Can you think of one of the rules of exponents that we have met before?

Why does this rule work? Why is it true?

Can you find another way of explaining it or justifying it?

Which way do you find easier/harder? Why?

Ask students what they learned by looking at other students' work and whether or not this helped them with expressions they had found difficult to find equivalences for or were unsure about:

Which expressions were most difficult to match? Why was this?

What changes did you make to your poster after looking at the poster from the other group?

Draw out any issues you noticed as students worked on the activity, making specific reference to any misconceptions you noticed. You may want to use the questions in the *Common issues* table to support your discussion.

Follow-up lesson: reviewing the assessment task (15 minutes)

Give each student a copy of the review task, *Properties of Exponents (Revisited)*, and their original scripts from the assessment task, *Properties of Exponents*. If you have not added questions to individual pieces of work then write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

Look at your original responses and the questions [on the board/written on your script.] Use what you have learnt to answer the questions.

Now look at the new task sheet, Properties of Exponents (Revisited). Can you use what you have learned to answer these questions?

Some teachers give this as homework.

SOLUTIONS

Assessment Task: *Properties of Exponents*

Correct answer	Common errors	General laws
1a). $2+2+2+2=2^3$.	Many write 2^4. Note that the expressions $2+2+2+2$ and $2\times2\times2\times2$ are commonly confused.	
b). $2\times2\times2\times2=2^4$.		$a\times a\times a\times.......a\ (n\ terms)\ = a^n$
c). 2^3 x $2^3 = 2^6$.	2^2 x $2^3 = 2^6$ is a common answer, multiplying exponents inappropriately.	$a^m a^n = a^{m+n}$
d). 2^3 x $3^3 = 6^3$.	6^6 is common, adding exponents inappropriately. Others may write 6^9; treating the exponents as a multiplier: (2 x 3) x (3 x 3) = 6 x 9.	$a^m b^m = (ab)^m$
e). $4^3 = 2^6$	$4^3 = (2^2)^3 = 2^5$ is a possible error. Note some may get this question correct for the wrong reason: 4 x 3 = 2 x 6.	$a^n = \left(\sqrt{a}\right)^{2n}$
f). $(3^2)^3 = 3^6$.	$(3^3)^3 = 3^6$ is a possible error (adding powers inappropriately).	This statement is a special case of $\left(a^m\right)^n = a^{mn}$.
g). $5^6 \div 5^2 = 5^4$.	$5^6 \div 5^2 = 5^3$ is a common error (dividing powers inappropriately)	$a^m \div a^n = a^{m-n}$
h). $5^2 - 3^2 = 2^4$	Some will write 2^0 by subtracting inappropriately	
i). $3^5 \div 3^6 = 3^{-1} = \frac{1}{3}$	Some students will have difficulties with the negative	$a^m \div a^n = a^{m-n}$

	exponent here.	

2. The correct order is (smallest first): 0^6 7^{-1} 6^0 2^3 3^2

Look for student reasoning where, for example, the order given is
7^{-1} $0^6, 6^0$ *(either order)* $2^3, 3^2$ *(either order)*
Students may be reasoning that the numbers are equivalent to -7, 0 (both cases), 6 (both cases).

Lesson Task: Expressions and Single Exponents

Each row displays a set of matching cards.

Expressions		Single Exponents	
E12 $2^3 \times 2^3$	E10 $(2^3)^2$	S6 2^6	S10 4^3
E2 $3^2 - 2^3$		S5 2^0	
E7 $3^2 + 3^3$	E1 $2^2 \times 3^2$	S8 6^2	
E4 $2^2 \div 2^3$		S4 2^{-1}	
E6 $2^2 - 2^2$		S9 0^2	
E8 $4^2 \div 2^3$		S1 2^1	
E9 $2^3 \div 2^{-2}$		S2 2^5	
E3 $2^2 + 2^3$	E11 3×2^2		
E13 $5^2 - 3^3$		S3 $(-2)^1$	
E14	E5	S7	

$(3^2 \times 2^2)^2$	$6^8 \div 6^4$	6^4

Assessment Task: *Properties of Exponents (Revisited)*

1a). $2 \times 2 \times 2 = 2^3$

b). $3 + 3 + 3 = 3^2$

c). $6^2 \times 6^4 = 6^6$

d). $3^3 \times 4^3 = 12^3$

e). $4^5 = 2^{10}$

f). $(6^2)^4 = 6^8$

g). $10^6 \div 10^3 = 10^3$

h). $10^2 - 6^2 = 4^3$

i). $4^5 \div 4^7 = 4^{-2} = \frac{1}{16}$

2). The correct order is (greatest first): 2^5 5^2 10^0 11^{-1} 0^{10}

Properties of Exponents

1. In each of the following questions write the missing exponents on the dotted lines.
   Show your reasoning in the spaces provided on the right.

a)	$2+2+2+2=2^{\cdots}$	
b)	$2\times 2\times 2\times 2=2^{\cdots}$	
c)	$2^{\cdots}\times 2^{3}=2^{6}$	
d)	$2^{3}\times 3^{3}=6^{\cdots}$	
e)	$4^{3}=2^{\cdots}$	
f)	$(3^{\cdots})^{3}=3^{6}$	
g)	$5^{6}\div 5^{2}=5^{\cdots}$	
h)	$5^{2}-3^{2}=2^{\cdots}$	
i)	$3^{5}\div 3^{\cdots}=3^{\cdots}=\frac{1}{3}$	

2. Write these five numbers in order of size, from smallest to greatest:

6^{0} $\quad 0^{6}$ $\quad 3^{2}$ $\quad 2^{3}$ $\quad 7^{-1}$

Smallest Greatest

Show your reasoning here:

Card Set: Expressions

E1 $2^2 \times 3^2$	E2 $3^2 - 2^3$
E3 $2^2 + 2^3$	E4 $2^2 \div 2^3$
E5 $6^8 \div 6^4$	E6 $2^2 - 2^2$
E7 $3^2 + 3^3$	E8 $4^2 \div 2^3$
E9 $2^3 \div 2^{-2}$	E10 $(2^3)^2$
E11 3×2^2	E12 $2^3 \times 2^3$
E13 $5^2 - 3^3$	E14 $(3^2 \times 2^2)^2$

Card Set: Single Exponents

S1 2^1	S2 2^5
S3 $(-2)^1$	S4 2^{-1}
S5 2^0	S6 2^6
S7 6^4	S8 6^2
S9 0^2	S10 4^3

Spare cards

Properties of Exponents (Revisited)

1. In each of the following questions write the missing exponents on the dotted lines.
   Show your reasoning in the spaces provided on the right.

a)	$2 \times 2 \times 2 = 2^{\cdots}$	
b)	$3 + 3 + 3 = 3^{\cdots}$	
c)	$6^{\cdots} \times 6^4 = 6^6$	
d)	$3^3 \times 4^3 = 12^{\cdots}$	
e)	$4^5 = 2^{\cdots}$	
f)	$(6^{\cdots})^4 = 6^8$	
g)	$10^6 \div 10^3 = 10^{\cdots}$	
h)	$10^2 - 6^2 = 4^{\cdots}$	
i)	$4^5 \div 4^{\cdots} = 4^{\cdots} = \dfrac{1}{16}$	

2. Write these five numbers in order of size, from greatest to smallest:

11^{-1} 10^0 0^{10} 5^2 2^5

Greatest Smallest

Show your reasoning here:

Powers of 2

A: $8 \times 4 = 32$

B: $16 \div 8 = 2$

C: $8 \div 16 = \frac{1}{2}$

D: $8 \div 8 = 1$

Working Together

Take it in turns to:

1. Select an expression card and find **all other cards** that have the same value as the one you have chosen.
2. Explain your matching to your partner.
3. Your partner must check your matching and challenge your explanation if they disagree.
4. Once agreed, glue the cards onto the poster and record your explanation for each match.
5. Continue to take turns until you have ten groups of cards.

Sharing Work

One person in your group:

- Write down your card matches on your mini-whiteboard.
- Go to another group's desk and compare your work with theirs.
- If there are differences in your matches, ask for an explanation.
- If you still don't agree, explain your own thinking.

The other person(s) should:

- Stay at your desk, and be ready to explain the reasons for your group's decisions.

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
Shell Center Team at the Center for Research in Mathematical Education
University of Nottingham, England:

Malcolm Swan,
Nichola Clarke, Clare Dawson, Sheila Evans, Colin Foster, and **Marie Joubert**
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and **Daniel Pead**

We are grateful to the many teachers and students, in the UK and the US, who took part in the classroom trials that played a critical role in developing these materials

The classroom observation teams in the US were led by
David Foster, Mary Bouck, and **Diane Schaefer**

This project was conceived and directed for
The Mathematics Assessment Resource Service (MARS) by
Alan Schoenfeld at the University of California, Berkeley, and
Hugh Burkhardt, Daniel Pead, and **Malcolm Swan** at the University of Nottingham

Thanks also to Mat Crosier, Anne Floyde, Michael Galan, Judith Mills, Nick Orchard, and Alvaro Villanueva who contributed to the design and production of these materials

This development would not have been possible without the support of
Bill & Melinda Gates Foundation

We are particularly grateful to
Carina Wong, Melissa Chabran, and **Jamie McKee**

The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org