

Mathematics Assessment Project
CLASSROOM CHALLENGES
A Formative Assessment Lesson

Solving Linear Equations in One Variable

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

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Solving Linear Equations in One Variable

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Solve linear equations in one variable with rational number coefficients.
- Collect like terms.
- Expand expressions using the distributive property.
- Categorize linear equations in one variable as having one, none, or infinitely many solutions.

It also aims to encourage discussion on some common misconceptions about algebra.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

8.EE: Analyze and solve linear equations and pairs of simultaneous linear equations.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 6 and 8:

1. Make sense of problems and persevere in solving them.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

This lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task that is designed to reveal their current understanding and difficulties. You then review their responses and create questions for students to consider when improving their work.
- After a whole-class introduction, students work in small groups on a collaborative discussion task, categorizing equations based on the number of solutions. Throughout their work, students justify and explain their thinking and reasoning.
- In the same small groups, students critique the work of others and then discuss as a whole-class what they have learned.
- In a follow-up lesson, students return to their original task and try to improve their own, individual responses.

MATERIALS REQUIRED

- Each student will need two copies of the assessment task *When are the equations true?*, a mini-whiteboard, a pen, and an eraser and a copy of *When are the equations true? (revisited)*.
- Each small group of students will need *Card Set: Equations*, a pair of scissors, a pencil, a marker, a glue stick, and a large sheet of paper for making a poster.
- There is a projector resource to support the whole-class introduction.

TIME NEEDED

15 minutes before the lesson for the assessment task, a 70-minute lesson (or two shorter ones), and 10 minutes in a follow-up lesson. Timings are approximate. Exact timings will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *When are the equations true?* (15 minutes)

Ask students to complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.

Give each student a copy of the assessment task.

Spend 15 minutes working individually, answering these questions. Make sure you explain your answers really clearly.

It is important that, as far as possible, students are allowed to answer the questions without assistance.

Students should not worry too much if they cannot understand or do everything because, in the next lesson, they will work on a similar task that should help them. Explain to students that, by the end of the next lesson, they should be able to answer questions such as these confidently. This is their goal.

Assessing students' responses

Collect students' responses to the task, and make some notes on what their work reveals about their current levels of understanding. The purpose of doing this is to forewarn you of the difficulties students will experience during the lesson itself, so that you may prepare carefully.

We suggest that you do not write scores on students' work. Research shows that this is counterproductive, as it encourages students to compare scores, and distracts their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by asking questions that focus their attention on aspects of their work. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight appropriate questions for each student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students, and write these on the board when you return the work to the students in the follow-up lesson.

When are the equations true?

1. Amy and Ben are trying to decide when the following equation is true:

$$5 - x = 6$$

They decide to compare their work.



Amy

$5 - x = 6$
 so $x = 6 - 5 = 1$
 so it is true when $x = 1$



Ben

If you take a number away from 5 the answer will be less than 5 so its never true.

Are Amy and Ben correct?
If not, where have they gone wrong?

Amy:

Ben:

What is your answer to the question?
.....

2. Amy and Ben now try to decide when the following equation is true:

$$8x - 6 = 2x$$

Comment on their work and identify any mistakes they have made.

Amy's work:

$8x$ and 6 are not "like terms"
If the equation was $8x - 6x = 2x$
then it would be always true

Ben's work:

When $x = 0$ $0 - 6 \neq 0$
 When $x = 1$ $8 - 6 = 2$ ✓
 When $x = 2$ $16 - 6 \neq 4$
 It doesn't work for all values of x , just for some.

What is your answer to the question?
.....

Common issues:**Suggested questions and prompts:**

<p>Assumes that subtraction is commutative</p> <p>For example: The student assumes that $5 - x = 6$ is the same as $x - 5 = 6$ and gives a value of $x = 11$ (Q1).</p>	<ul style="list-style-type: none"> • Is $3 - 2$ the same as $2 - 3$? Try some other numbers. Will it ever be the same? • Now look at your work. Is $5 - x$ the same as $x - 5$? How do you know? • Check your work by substituting $x = 11$ back into the equation. What do you notice?
<p>Considers positive numbers only</p> <p>For example: The student assumes that Ben is correct (Q1).</p>	<ul style="list-style-type: none"> • Consider this: 3 subtract 2 equals 1; 3 subtract 1 equals 2; 3 subtract 0 equals 3. Can you see any patterns? What comes next? Three subtract something equals four?
<p>Fails to explain why Amy and Ben are incorrect (Q1)</p>	<ul style="list-style-type: none"> • How can you show that Amy's value of x does not satisfy the equation? • How could you convince someone else that Amy/Ben has made a mistake? How could you show this?
<p>Understands both Amy and Ben are incorrect, but does not provide a correct solution</p>	<ul style="list-style-type: none"> • What math can you use to work out if there is a value for x that satisfies the equation? [Guess and check or make x the subject of the equation.]
<p>Incorrect use of the equal sign</p> <p>For example: The student writes $5 - x = 6 = x = -1$.</p>	<ul style="list-style-type: none"> • Carefully check your work. In your head say the math you've written. Does it make sense if you read it from left to right?
<p>Solves the equation to give $x = -1$ correctly and states the equation is true</p>	<ul style="list-style-type: none"> • Is this equation always true? • How many solutions does this equation have? • How do you know?
<p>Assumes Amy is correct (Q2)</p>	<ul style="list-style-type: none"> • Give x a value. Are the terms alike now? Can you now subtract 6 from $8x$? • Can you find a value for x that makes the equation true?
<p>Assumes Ben is correct (Q2)</p>	<ul style="list-style-type: none"> • How many solutions does an equation need to have, to be true? • Is $x = 1$ the only solution to this equation? How do you know?
<p>Completes the task</p>	<ul style="list-style-type: none"> • Can you think of a different way of showing when the equation is true? Which method is most convincing? Why do you think this is? • Can you write down a new equation, that is always/never true? How could you prove this? • Can you think of an equation that is sometimes true, but has more than one solution?

SUGGESTED LESSON OUTLINE

Whole-class introduction (15 minutes)

Give each student a mini-whiteboard, a pen, and an eraser. Maximize participation in the discussion by asking all students to show you their solutions on their mini-whiteboards.

This introduction will provide students with a model of how they should justify their solutions in the collaborative activity.

Display Slide P-1 of the projector resource:

True or False?

$$4x + 1 = 3$$

Can you give me a value for x that makes this equation **false**?

Show the calculations that explain your answer.

Students should not have any problems with finding a suitable value for x , but may not be too adventurous in their choices. Spend some time discussing the values given and the reasons for each choice, identifying any common choices, as well as any calculation errors.

Display Slide P-2 of the projector resource:

True or False?

$$4x + 1 = 3$$

Can you give me a value for x that makes this equation **true**?

Show the calculations that explain your answer.

Students may struggle with this at first, especially if their chosen method is substituting values for x . Encourage students to explore fractions, decimals, and negative numbers as well as positive whole numbers.

If students find a value for x , challenge them to consider if there are any other values of x . They should be encouraged to justify why this is the only value for x that makes the equation true and how they can be sure of this.

Would we describe this equation as always true, never true or sometimes true? [Sometimes true.]

When is it true? [When $x = \frac{1}{2}$.]

Are there any other values for x that make the equation true? How do you know?

If students struggle to justify why there is only one value for x that makes the equation true, then show Slide P-3 of the projector resource:

How many different values of x make the equation true?

$$4x + 1 = 3$$

Cheryl:

$x = 2$
 $4 \times 2 + 1 = 9 \neq 3$ (not 3)

$x = 1$ $x = 0$
 $4 \times 1 + 1 = 5 \neq 3$ $4 \times 0 + 1 = 1 \neq 3$

$x = \frac{1}{2}$
 $4 \times \frac{1}{2} + 1 = 3$ ✓

There is only one value for x that makes the equation true.

Stacey:

$4x + 1 = 3$

This means $4x = 2$ and this always has to be true.

To make 2 x must be $\frac{1}{2}$ because $4 \times \frac{1}{2} = 2$

x can't be any other value.

Ask students to vote on which explanation they prefer. Ask one or two students to justify their preference.

Cheryl has used the method of guess and check. Once she has found a value for x that works, she has assumed that this is the only value and so has not tried to find any others. She has not explained why this is the case.

Stacey has found the value of x as $\frac{1}{2}$ by ‘undoing’ the equation. She has stated that x cannot be any other value, using her knowledge that the only number to give an answer of 2 when multiplied by 4 is $\frac{1}{2}$. She has not used a formal algebraic proof.

Collaborative activity: Always, Sometimes, or Never True? (30 minutes)

Ask students to work in groups of two or three.

Give each group *Card Set: Equations*, a pair of scissors, a large sheet of paper, a marker, and a glue stick. Make sure that each group has a pencil to use when writing explanations.

Explain to students that during this lesson they will consider a number of equations in the same way as they have just been doing. They will work in small groups to produce a poster that will show each equation classified according to whether it is always, sometimes or never true.

Ask students to divide their large sheet of paper into three columns and head separate columns with the words: ‘Always True’, ‘Sometimes True’, ‘Never True’.

What does always true mean? [The equation is true for any value of x .]

What does never true mean? [There are no values of x that make the equation true.]

How many examples does it take to prove that an equation is sometimes true? [Two. One value for x that makes the equation true, and one value for x that makes the equation false.]

All of the equations are linear, so there will only ever be one solution for the equations that are sometimes true. Students may not be aware of this and should be encouraged to explore the number of solutions for different equations as appropriate.

Now explain how students are to work together:

One partner, select an equation, cut it out and place it in one of the columns, explaining why you choose to put it there.

If you think the statement is sometimes true, give values of x for which it is true. If you think the equation is always true or never true, explain how you can be sure this is the case.

*Your partner should then either challenge the explanation if they disagree, or if they agree, describe it in their own words. Once you all agree, stick the statement card on the poster and write an explanation next to the card. Explanations should be written **in pencil**.*

Swap roles and continue to take turns until all the equations have been placed.

Slide P-4 of the projector resource summarizes these instructions.

The purpose of this structured work is to encourage each student to engage with their partner's explanations, and to take responsibility for their partner's understanding. Students should use their mini-whiteboards for calculations and to explain their thinking to each other.

It does not matter if students do not manage to place all the cards. It is more important that everyone in each group understands the categorization of each equation.

While students are working in small groups you have two tasks: to make a note of student approaches to the task, and to support student reasoning.

Make a note of student approaches to the task

Listen and watch students carefully. In particular, listen to see whether students are addressing the difficulties they experienced in the assessment task. Are students struggling with fractions? Do students justify their categorization by using the commutative or distributive properties? Are students using guess and check, or do they make x the subject of the equation? Do students have a systematic strategy? Do students have problems with unlike terms? Do students misuse the equal sign? For example, students may write $3x - 5 = 2x = x = 5$.

You may want to use the questions in the *Common issues* table to help address misconceptions.

Support student reasoning

Encourage students to explain their reasoning carefully.

You have shown the statement is true for this specific value of x . Now convince me it is always true for every number!

Can you use algebra to justify your decision for this card?

You say this equation is never true. How do you know for sure?

What mathematical reasoning can you give for why this equation is true/not true?

Encourage students to write an explanation for the placement of each card.

If one student has placed an equation, challenge another student in the group to provide an explanation in their own words.

Drew placed this equation. Ellie, why does Drew think the equation goes here?

If the student is unable to answer this question, ask the group to discuss the work further. Explain that you will return in a few minutes to ask a similar question.

If students are struggling to get started, select one of the equations or make up a different equation of your own and work on it together, modeling the categorization process. Ask questions that help students to clarify their thinking:

When can you make this equation true?

Can you show me an example of when it isn't true?

Does this look like something you know?

What are you thinking?

Which column does this equation belong to? How do you know?

You may find it helpful to model the expectations for the task to the whole class rather than just groups that are struggling to get started. If this is the case, using a different equation to those on the *Card Set: Equations* may be appropriate.

If students finish the task quickly, ask them to create new examples and add them to the table.

Sharing posters (10 minutes)

Once students have completed their posters, they critique each others' work.

You are now going to critique each others' posters.

In your groups, move to another table and look at their poster. If you disagree with where an equation has been placed, put a circle around the equation and write three things:

Why you disagree.

Which column you think the equation needs moving to.

Why you think the equation belongs there.

Put a circle around your comments and write your initials next to them. Use a pen to do this.

After a few minutes you will move on to the next poster and critique it in the same way.

Slide P-5 of the projector resource summarizes these instructions.

When a poster is being critiqued for the second or third time, students may find that they disagree with a previous comment. If this is the case, ask students to write their reason for the disagreement on the poster, next to the previous comment.

When students have had a chance to critique a couple of posters from different groups, ask them to return to their own poster and read the comments that have been left.

Whole-class discussion (15 minutes)

Organize a whole-class discussion about what has been learned and explore the different methods of justification used when categorizing equations. Try to include a discussion of at least one equation from each column.

Give me an equation that is always true/sometimes true/never true.

Why did you put this equation in this column?

Did anyone put this equation in a different column?

You may want to, first, select a card that most groups placed correctly as this may encourage good explanations. Once one group has justified their choice for a particular equation, ask other students to contribute ideas of alternative approaches, and their views on which reasoning method was easier to follow.

Which column did you put this equation in? Can you explain your decision?

Can anyone improve this explanation?

Does anyone have a different explanation?

Which explanation do you prefer? Why?

Ask students what they learned by looking at other students' work and whether or not this helped them with equations that they had found difficult to categorize or were unsure about:

*Which equation did you find the most difficult to categorize? Why do you think this was?
Did seeing where another group had placed this equation help? If so, in what way did it help?
In what ways did having another group critique your poster help?*

Did looking at another group's poster help you with your own work? Can you give an example?

During the discussion, draw out any issues you have noticed as students worked on the activity, making specific reference to the misconceptions you noticed. You may want to use the questions in the *Common issues* table to support the discussion.

At this stage, you could also check that students know some key terms such as 'solution' and 'identity':

Can you tell me another name for an equation that is always true? [Identity.]

What do we call the value of x for which an equation is true? [The solution.]

Follow-up lesson: improving individual solutions to the assessment task (10 minutes)

Return to students their original assessment, as well as a second, blank copy of the task.

If you have not added questions to individual pieces of work, write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

Look at your original responses and read through the questions I have written.

Spend a few minutes thinking about how you could improve your work.

Using what you have learned, have another go at the task on the blank copy provided.

Alternatively, you may like to give students a copy of *When are the equations true? (revisited)*. This sheet provides some further questions that will help you to check on what students have learned.

SOLUTIONS

Assessment task: When are the equations true?

1. The equation $5 - x = 6$ is only true when $x = -1$.

Amy has solved the equation $5 + x = 6$ instead of $5 - x = 6$.

The second line should read: $-x = 6 - 5$ so $x = -1$.

Ben has not considered the effects of subtracting a negative number.

He has not tried to substitute values for x or shown any algebraic manipulation.

2. The equation $8x - 6 = 2x$ is only true when $x = 1$.

Amy has made a valid point about like terms, but has not looked beyond this to see if there is a value of x that makes this equation true.

Ben has found a solution for the equation using substitution.

He has not proved that only one value for x satisfies the equation. In order to do this he needs to solve the equation.

Collaborative Activity:

Always true	Sometimes true	Never true
E2 $3 + x = x + 3$	E1 $2 - x = x - 2$ True when $x = 2$.	E3 $x + 5 = x - 3$
E8 $7x + 14 = 7(x + 2)$	E7 $6x = x$ True when $x = 0$.	E11 $5x - 5 = 5(x + 1)$
E10 $\frac{2x + 4}{2} = x + 2$	E12 $4x = 4$ True when $x = 1$.	E6 $2(x + 1) = 2x + 1$
	E9 $\frac{10}{2x} = 5$ True when $x = 1$.	
	E5 $\frac{x}{2} = x$ True when $x = 0$.	
	E4 $3x - 5 = 2x$ True when $x = 5$.	

Assessment task: When are the equations true? (revisited)

	Equation	For what values of x is it true?
1.	$12 - x = 15$	$12 - x = 15$ $\Leftrightarrow -x = 3$ $\Leftrightarrow x = -3$ It is only true when $x = -3$.
2.	$x - 3 = 3 - x$	$x - 3 = 3 - x$ $\Leftrightarrow 2x = 6$ $\Leftrightarrow x = 3$ It is only true when $x = 3$
3.	$\frac{x}{2} = 6$	$\frac{x}{2} = 6$ $\Leftrightarrow x = 12$ It is only true when $x = 12$
4.	$\frac{10}{x} = 20$	$\frac{10}{x} = 20$ $\Leftrightarrow 10 = 20x$ $\Leftrightarrow x = \frac{1}{2}$ It is only true when $x = 1/2$
5.	$3(x + 4) = 3x + 4$	$3(x + 4) = 3x + 4$ $\Leftrightarrow 3x + 12 = 3x + 4$ $\Leftrightarrow 12 = 4$ This final statement is false, so the equation is never true.
6.	$2(x + 3) = 2x + 6$	$2(x + 3) = 2x + 6$ $\Leftrightarrow 0 = 0$ This final statement is always true, so the equation is always true and is an identity.

When are the equations true?

1. Amy and Ben are trying to decide when the following equation is true:

$$5 - x = 6$$

They decide to compare their work.



Amy

$$5 - x = 6$$

$$\text{so } x = 6 - 5 = 1$$

so it is true when $x = 1$



Ben

If you take a number away from 5
the answer will be less than 5
So it's never true.

Are Amy and Ben correct?

If not, where have they gone wrong?

Amy:

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.....
.....

Ben:

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What is your answer to the question?

.....
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.....

2. Amy and Ben now try to decide when the following equation is true:

$$8x - 6 = 2x$$

Comment on their work and identify any mistakes they have made.

Amy's work:

*8x and 6 are not "like terms"
If the equation was $8x - 6x = 2x$
then it would be always true*

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Ben's work:

*When $x=0$ $0-6 \neq 0$
When $x=1$ $8-6 = 2 \checkmark$
When $x=2$ $16-6 \neq 4$
It doesn't work for all
values of x , just for some.*

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What is your answer to the question?

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Card Set: Equations

E1 $2 - x = x - 2$	E2 $3 + x = x + 3$
E3 $x + 5 = x - 3$	E4 $3x - 5 = 2x$
E5 $\frac{x}{2} = x$	E6 $2(x + 1) = 2x + 1$
E7 $6x = x$	E8 $7x + 14 = 7(x + 2)$
E9 $\frac{10}{2x} = 5$	E10 $\frac{2x + 4}{2} = x + 2$
E11 $5x - 5 = 5(x + 1)$	E12 $4x = 4$

When are the equations true? (revisited)

1. Try to decide when the following equations are true.
The first one has been done as an example.

	Equation	For what values of x is it true?
	$6x + 3 = 15$	<i>This is only true when $x = 2$.</i>
1.	$12 - x = 15$	
2.	$x - 3 = 3 - x$	
3.	$\frac{x}{2} = 6$	
4.	$\frac{10}{x} = 20$	
5.	$3(x + 4) = 3x + 4$	
6.	$2(x + 3) = 2x + 6$	

True or False?

$$4x + 1 = 3$$

Can you give me a value for x that makes this equation **false**?

Show the calculations that explain your answer.

True or False?

$$4x + 1 = 3$$

Can you give me a value for x that makes this equation **true**?

Show the calculations that explain your answer.

How many different values of x make the equation true?

$$4x + 1 = 3$$

Cheryl:

$$x = 2$$

$$4 \times 2 + 1 = 9 \quad \times \quad (\text{not } 3)$$

$$x = 1$$

$$4 \times 1 + 1 = 5 \quad \times$$

$$x = 0$$

$$4 \times 0 + 1 = 1 \quad \times$$

$$x = \frac{1}{2}$$

$$4 \times \frac{1}{2} + 1 = 3 \quad \checkmark$$

There is only one value for x that makes the equation true.

Stacey:

$$4x + 1 = 3$$

This means $4x = 2$ and this always has to be true.

To make 2 x must be a $\frac{1}{2}$ because $4 \times \frac{1}{2} = 2$

x can't be any other value.

Always, Sometimes or Never True?

1. One partner, select an equation, cut it out and place in one of the columns, explaining why you chose to put it there.
2. If you think the equation is **sometimes** true, give values of x for which it is true and for which it is false.
If you think the equation is **always** true or **never** true, explain how you can be sure this is the case.
3. Partners challenge the explanation if you disagree or describe it in their own words if they agree.
4. Once agreed, stick the equation card on the poster and write an explanation on the poster **in pencil** next to the card.
5. Swap roles and continue to take turns until all equations are placed.

Sharing Posters

1. Move to another table and look at their poster.
2. If you disagree with where an equation has been placed, put a circle around the equation and write **in pen**:
 - Why you disagree.
 - Which column you think the equation needs moving to.
 - Why you think the equation belongs there.
3. Circle your comments and write your initials next to them.

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
Shell Center Team at the Center for Research in Mathematical Education
University of Nottingham, England:

Malcolm Swan,
Nichola Clarke, Clare Dawson, Sheila Evans, Colin Foster, and Marie Joubert
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and Daniel Pead

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The full collection of Mathematics Assessment Project materials is available from

<http://map.mathshell.org>