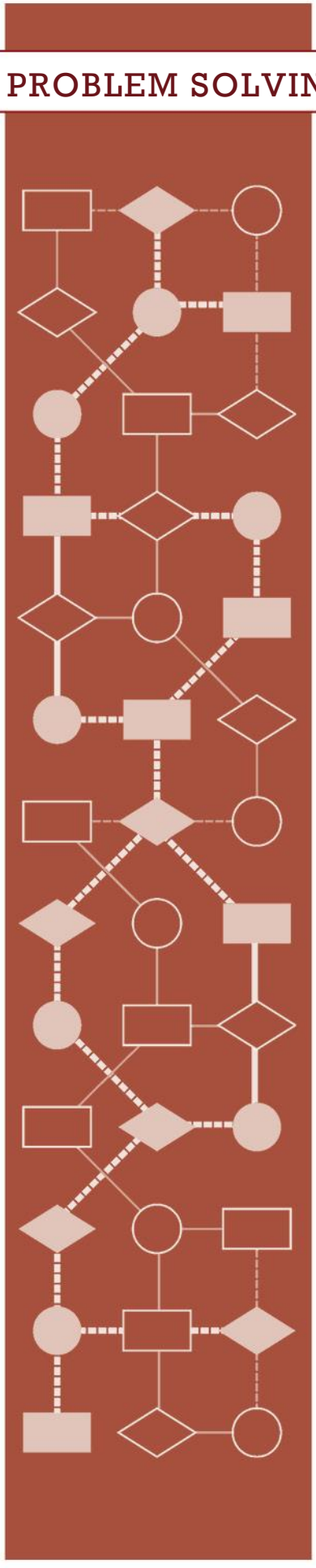
PROBLEM SOLVING



Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Modeling:
Making Matchsticks

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Modeling: *Making Matchsticks*

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Interpret a situation and represent the variables mathematically.
- Select appropriate mathematical methods.
- Interpret and evaluate the data generated.
- Communicate their reasoning clearly.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with particular emphasis on Practices 1, 2, 3, 4, and 6.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

8.G: Solve real-world and mathematical problems involving volumes of cylinders, cones, and spheres.

INTRODUCTION

- Before the lesson, students tackle the problem individually. You assess their responses and formulate questions that will prompt them to review their work.
- At the start of the lesson, students respond individually to the questions set, and then work in groups to combine their thinking and produce a collaborative solution in the form of a poster.
- In the same small groups, students evaluate and comment on some sample responses. They identify the strengths and mistakes in these responses and compare them with their own work.
- In a whole-class discussion, students explain and compare the solution strategies they have seen and used.
- In a follow-up lesson, students reflect on their work and their learning.

MATERIALS REQUIRED

- Each student will need a calculator, a copy of *Making Matchsticks,* a copy of the *Formula Sheet*, a blank sheet of paper, and a copy of the *How Did You Work?* questionnaire.
- Each small group of students will need a large sheet of paper for making a poster, felt-tipped pens, and copies of *Sample Responses to Discuss*.
- There is a projector resource to help you with whole-class discussions.

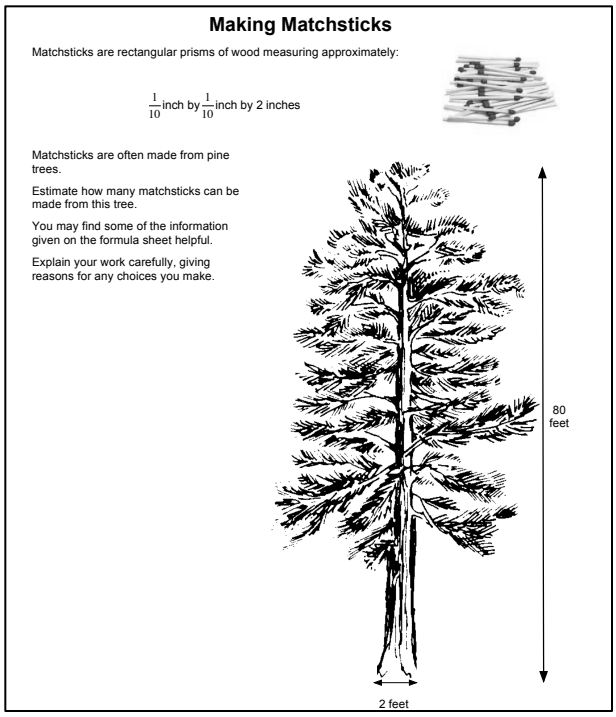
TIME NEEDED

15 minutes before the lesson, a 90-minute lesson (or split into two shorter lessons), and 15 minutes in a follow-up lesson. Timings given are only approximate. Exact timings will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Making Matchsticks* (15 minutes)

Have the students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the next lesson.



Give out the *Making Matchsticks* task, a copy of the *Formula Sheet*, and a blank sheet of paper for students to work on.

> *This activity is about estimating how many matchsticks can be made from the wood in one pine tree.*
>
> *You have been given a formula sheet with some information on which may be helpful to you.*
>
> *Read through the task carefully and remember to show all your work, so that I can understand your reasoning.*
>
> *As well as trying to solve the problem, try to present your work in an organized and clear way.*

It is important that, as far as possible, students are allowed to complete the task without your assistance.

Students who sit together often produce similar responses so that when they come to compare their work they have little to discuss. For this reason, we suggest that, when students do the task individually, you ask them to move to different seats. Then, at the beginning of the formative assessment lesson, allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given in the *Common issues* table on the pages T-3 and T-4. We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students at the beginning of the lesson.

Common issues:	Suggested questions and prompts:
Has difficulty getting started	• What do you know? What do you need to find out? • How could you simplify the problem?
Ignores the units For example: The student calculates the volume of a matchstick in cubic inches and the volume of the tree trunk in cubic feet.	• What measurements are given? • Does your answer seem reasonable if you consider the size of a matchstick compared to the size of a pine tree?
Makes incorrect assumptions For example: The student multiplies the volume of the tree trunk in cubic feet by 12 and assumes this gives the volume of the tree trunk in cubic inches.	• Can you explain why you have multiplied by 12? • When you figure out a volume how many dimensions do you multiply together? How does this calculation effect how you convert the volume from cubic feet to cubic inches? • Can you describe the dimensions of the tree in inches? What do you notice?
Uses an inappropriate formula For example: The student calculates the *surface area* of a rectangular prism from the dimensions given for the tree.	• Does your choice of formula make good use of all the wood in the tree trunk? • Is this the best model for a tree trunk? • What is the difference between area and volume?
Works unsystematically	• Would someone in your class who has not used this method be able to follow your work? • Can you describe your method as a series of logical steps?
Work is poorly presented For example: The student underlines numbers and it is left to the reader to work out why this is the answer as opposed to any other calculation.	• Can you explain each part of your solution? • What does each of these calculations represent? • Can you justify the choices you have made?
Difficulty substituting into a formula For example: The student multiplies the radius by 2, rather than squaring, when using the formula for the volume of a cone/cylinder. Or: The student substitutes diameter rather than radius into the formula for the volume of a cone/cylinder.	• What is the difference in meaning between $2r$ and r^2? • Does your answer seem reasonable? • How can you check your work against the information given in the problem?

Common issues:	Suggested questions and prompts:
Work is incomplete For example: The student does not divide the volume of the tree trunk by the volume of a matchstick.	• What do your calculations represent? • Have you found out how many matchsticks can be made from the tree?
Rounds to one or more decimal places	• Why won't part of a matchstick count in your estimate?
Completes the task	• How can you check that the method you have used has given a reasonable estimate? • Can you try a different method to check your answer? • What assumptions have you made?

SUGGESTED LESSON OUTLINE

Reviewing individual solutions to the problem (10 minutes)

Return the task to the students.

If you have not added questions to individual pieces of work, write your list of questions on the board. Students should select questions appropriate to their own work and spend a few minutes thinking about their responses to them.

> *Recall the Making Matchsticks problem. What was the task about?*
>
> *I have looked at your work and I have some questions about it.*
>
> *I would like you to think, on your own, about my questions and how your work could be improved.*

When a list of questions is written on the board rather than on individual pieces of work, some students may struggle to identify which questions they should be considering. If this is the case, it may be helpful to give students a printed version of the list of questions with the relevant ones highlighted.

Collaborative work: making posters (30 minutes)

Organize the class into small groups of two or three students and hand out poster paper and felt-tipped pens. Ask students to have another go at the task, but this time they should combine their ideas and use what they have learned from reviewing their individual solutions.

> *You each have your own individual solution and have been thinking about how you might improve these. Share your method with your partner(s) and your ideas for improving your work.*
>
> *Together in your group, agree on the best method for completing the problem and produce a poster that shows a joint solution to the task. Try to make this better than your individual solutions.*
>
> *State on your poster any assumptions you have made and give clear reasons for your choice of method.*

While students work in small groups you have two tasks: to support student problem solving and to note different student approaches to the task. You can then use this information to focus a whole-class discussion at the end of the collaborative work.

Slide P-1 of the projector resource, *Working Together*, summarizes these instructions.

Note different student approaches to the task.

In particular, note whether students' original methods are the same or different. If they are different, how do they decide which method to use for their joint solution? Or, do they use a totally different method altogether? What are their reasons for the choice of method? Some students may not take units of measure into consideration. If they do, how do they deal with the different units? Are there any common mistakes being made? What do they do if they get stuck? Are they checking their answers? Are they aware of any assumptions they have made?

Support student problem solving.

If students are struggling to produce a joint solution to the task, encourage them to identify the strengths and weaknesses of the methods employed in their individual responses. Can any of these methods be improved to produce a group solution that is better than the original individual response? Can they think of any other approaches to try?

Try not to make suggestions that move students towards a particular approach to the task. Instead, ask questions that help students to clarify their thinking. For example, you may ask them to consider these questions:

What have you done that you both [all] agree on?
What else do you need to find?
Have you used all the information given in the task?
What do you now know that you didn't know before?
Do your calculations make sense?
What assumptions have you made?

The purpose of these questions is to help students to track and review their problem solving strategies. They should be encouraged to give reasons for the choices they have made.

You may also want to use some of the questions in the *Common issues* table to support your own questioning or, if the whole class is struggling on the same issue, write relevant questions on the board and hold a brief whole-class discussion.

Sharing different approaches (10 minutes)

Hold a whole-class discussion on the methods used to produce a group solution. Ask two or three groups of students with contrasting methods to describe the method used and the ways in which this method differs from their initial individual responses. Did the students check their work? If they did, what checking method did they use?

Extending the lesson over two days

If you are taking two days to complete the unit then you may want to end the first lesson here. At the start of the second day, briefly remind students of their previous work before moving on to the collaborative analysis of sample responses.

Collaborative analysis of *Sample Responses to Discuss* (30 minutes)

Once students have had sufficient time to discuss some different approaches, give each group copies of the *Sample Responses to Discuss*.

This task gives students an opportunity to evaluate a variety of possible approaches to the task, without providing a complete solution strategy. Students should thoughtfully answer the questions below each piece of sample student work and be encouraged to think carefully about ways in which the work could be improved.

In your groups you are now going to look at some student work on the task. Notice in what ways this work is similar to yours and in which ways it is different.

There are some questions for you to answer as you look at the work. You may want to correct the work or add annotations to make it easier to follow.

It may not be appropriate, or there may not be enough time, for all students to analyze all three sample responses. Each response highlights different misconceptions so, depending on progress already made on the task, it may be appropriate to issue different sample responses to different groups. For example, groups that have struggled with the need to find volume could be given Jaabir's work, while groups that have ignored the difference in units could be given Chan's work. The whole-class discussion held after the collaborative work should help to inform your decision on whether or not to be selective about which sample responses students are given.

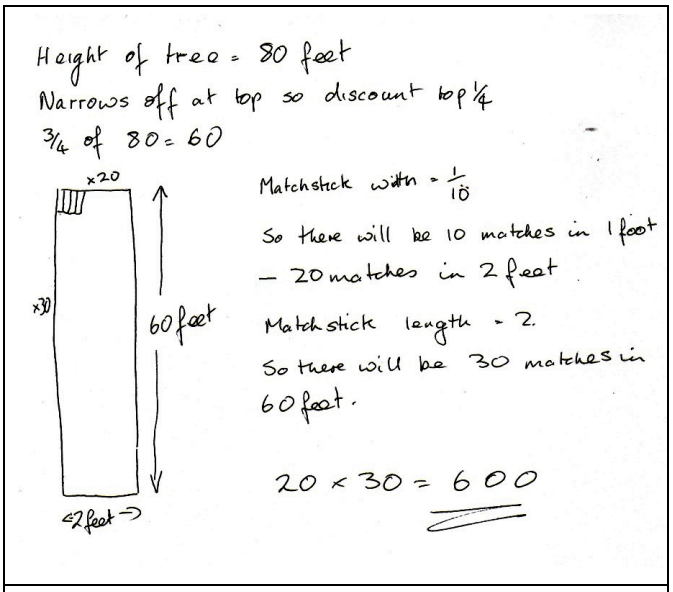
Slide P-2 of the projector resource, *Evaluating Sample Student Responses*, describes how students should work together.

Evaluating Sample Student Responses

1. Imagine you are the teacher and have to assess the student work.
2. Take turns to work through a student's solution.
   - Write your answers on your mini-whiteboards.
3. Explain your answer to the rest of the group.
4. Listen carefully to explanations.
   - Ask questions if you don't understand.
5. Once everyone is satisfied with the explanations, write the answers below the student's solution.
   - Make sure the student who writes the answers is not the student who explained them.

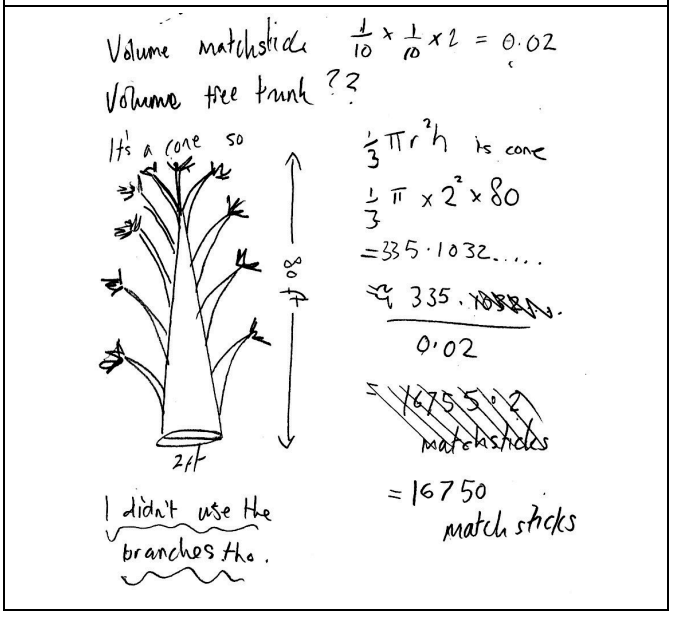
Jaabir has made the assumption that the tree trunk does not extend to the top of the tree. He has failed to see the task as a volume problem and has modeled the tree trunk as a rectangle, rather than a 3-dimensional object. When working out how many matchsticks will fit inside the rectangle, he has not taken into account the difference in units.

Jaabir has not checked to see if his answer makes sense.

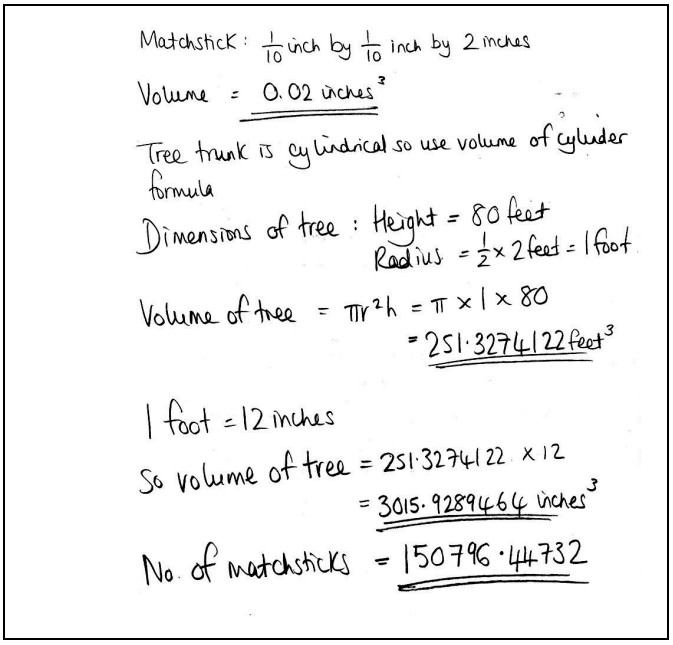


Chan has correctly calculated the volume of a matchstick in cubic inches and has attempted to find the volume of the tree trunk in cubic feet, but has used 2 feet as a measurement for the radius. There is evidence that he has thought about rounding and has rounded the volume of the cone prior to dividing by the volume of a matchstick, after an initial attempt, which led to a decimal solution. Chan has failed to take into account the different units.

Chan has not checked to see if his answer makes sense.



Sherida has correctly calculated the volumes of the matchstick and the tree trunk, and has noted their units. She multiplies the volume of the tree trunk by 12 (instead of 12^3) in what appears to be an attempt to deal with the difference in units. Her final solution is given to an unrealistic degree of accuracy.



Whole-class discussion (10 minutes)

Now hold a whole-class discussion to consider the different approaches used in the sample work. Focus the discussion on those parts of the small-group tasks students found difficult. Ask the students to compare the different solution methods.

Which approach did you like best? Why?

Which approach did you find most difficult to understand?

You may also want to compare students' own work with the sample student responses.

Did any group use a similar method to Jaabir, Chan or Sherida?

What was the same about the work? What was different about the work?

Did analyzing the responses enable anyone to see errors in their own work?

You may want to use Slides P-3, P-4 and P-5 of the projector resource and the questions in the *Common issues* table to support this discussion.

For this task, how accurate do the calculations need to be? [e.g. to the nearest 10 matches, to one decimal place etc.] Why?

What assumptions have the students' made? Are they reasonable? How do the assumptions impact on their answers?

Follow-up lesson: individual reflection (15 minutes)

Once students have had a chance to discuss the sample responses as a whole-class, distribute the *How Did You Work?* questionnaire. Ask students to spend a few minutes, individually, answering the questions.

Think carefully about your work in this lesson and the different methods you have seen and used.

The questionnaire should help students monitor and review their progress.

If you have time you may also want to ask your students to read through their original solution and using what they have learned, attempt the task again.

Some teachers give this as a homework task.

SOLUTIONS

There are a number of ways of modeling this problem, but solutions should ideally include the following:

- Students should model the task as a volume problem.
- Students should take into account the difference in units and address this at some point during the solution process.
- Any assumptions made should be clearly stated and explained.
- The estimate for the number of matchsticks should be given as a whole number, with evidence of thoughtful rounding as appropriate.
- A second model should be used to check whether or not the solution is reasonable.

The following table may be helpful when reviewing student work:

Volume of a matchstick = 0.02 cubic inches (or 0.00001157 cubic feet).

3D object	Volume (cubic feet)	Volume (cubic inches)	Number of matchsticks (nearest million)
Square based pyramid	106.7	184,320	9 million
Cone	83.7	144,765	7 million
Cylinder	251.2	434,294	22 million
Rectangular prism	320	552,960	28 million

The pyramid and cone both take into account the tapering of the tree and will possibly provide rather better estimates than the cylinder and rectangular prism, but this does depend on how students justify their work. 7 to 8 million would seem a reasonable estimate.

Making Matchsticks

Matchsticks are rectangular prisms of wood measuring approximately:

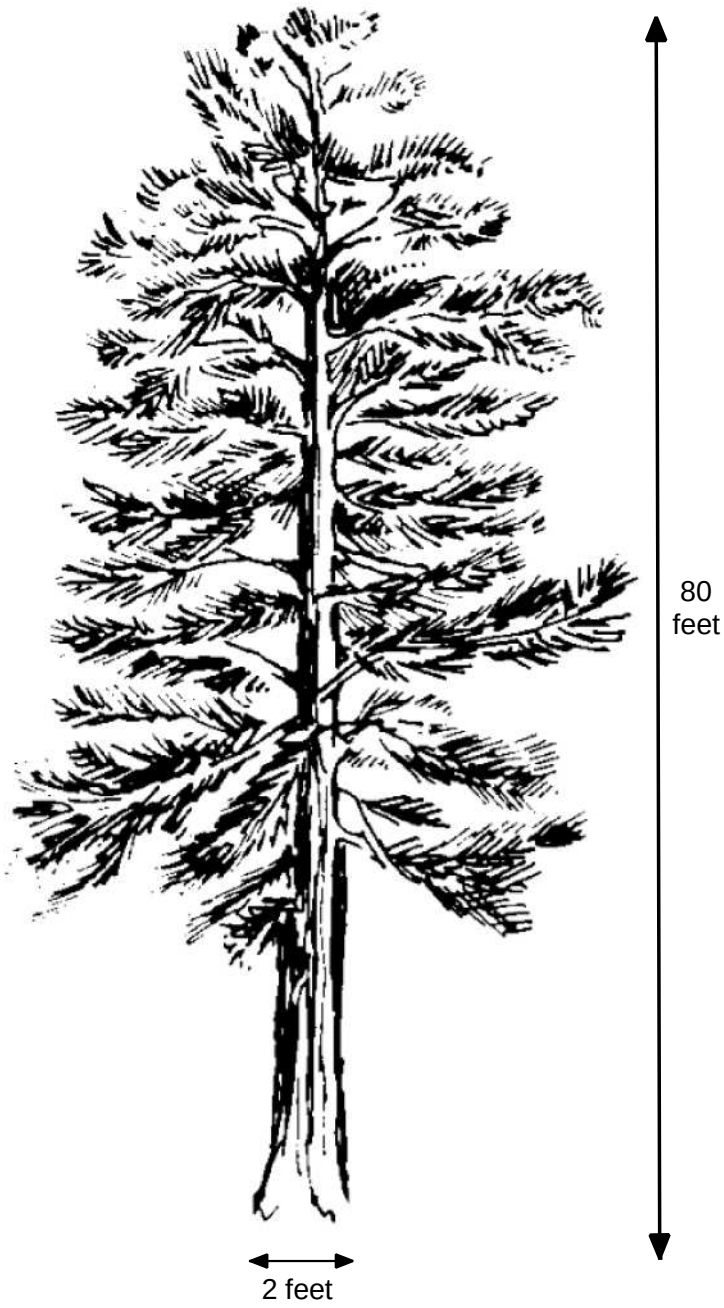
$\frac{1}{10}$ inch by $\frac{1}{10}$ inch by 2 inches

Matchsticks are often made from pine trees.

Estimate how many matchsticks can be made from this tree.

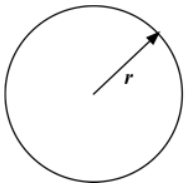
You may find some of the information given on the formula sheet helpful.

Explain your work carefully, giving reasons for any choices you make.

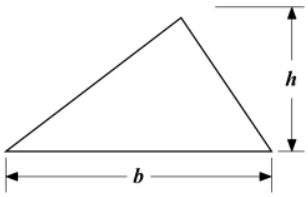


Formula Sheet

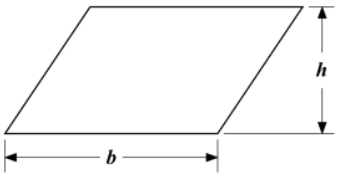
Area of a circle: πr^2



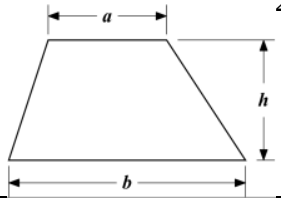
Area of a triangle: $\frac{bh}{2}$



Area of a parallelogram: bh

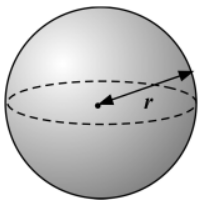


Area of a trapezoid: $\frac{1}{2}(a+b)h$

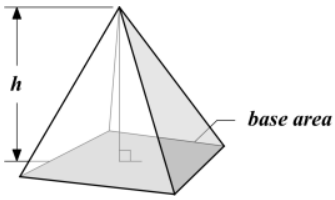


Surface area of a sphere: $4\pi r^2$

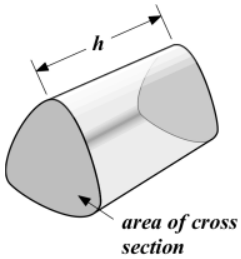
Volume of a sphere: $\frac{4}{3}\pi r^3$



Volume of a pyramid: $\frac{1}{3}$ *base area* $\times h$

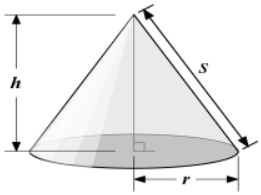


Volume of a prism: area of cross section $\times$ height



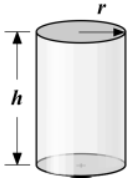
Volume of a cone: $\frac{1}{3}\pi r^2 h$

Curved surface area of cone: πrs



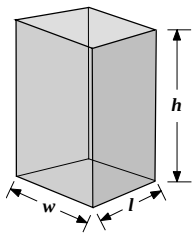
Volume of a cylinder: $\pi r^2 h$

Curved surface area of cylinder: $2\pi rh$

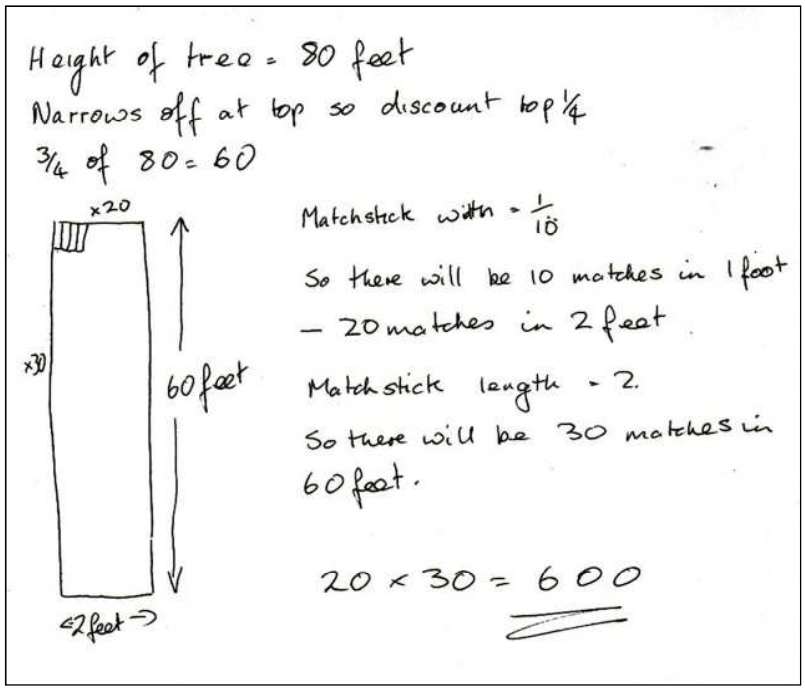


Volume of a rectangular prism: lwh

Surface area of rectangular prism: $2(wh+lh+wl)$



Sample Responses to Discuss: Jaabir



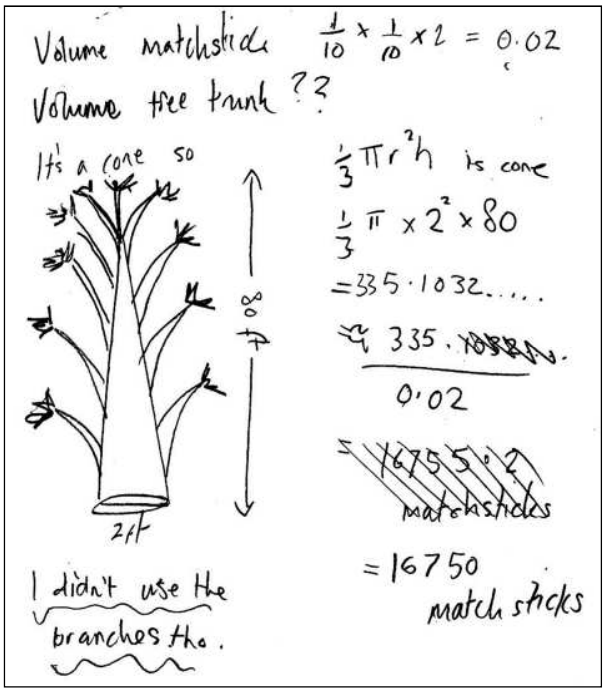
What has Jaabir calculated?

What mistakes has Jaabir made?

In what ways could Jaabir's work be improved?

To help you to understand Jaabir's work, what question(s) could you ask him?

Sample Responses to Discuss: Chan



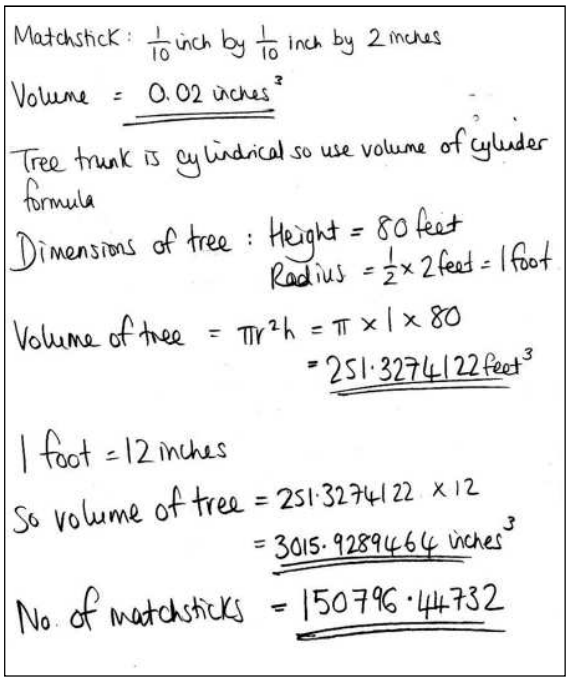
What has Chan done correctly?

What mistakes has Chan made?

In what ways could Chan's work be improved?

To help you to understand Chan's work, what question(s) could you ask him?

Sample Responses to Discuss: Sherida


Matchstick: $\frac{1}{10}$ inch by $\frac{1}{10}$ inch by 2 inches

Volume = 0.02 inches^3

Tree trunk is cylindrical so use volume of cylinder formula

Dimensions of tree: Height = 80 feet

Radius = $\frac{1}{2} \times 2$ feet = 1 foot

Volume of tree = $\pi r^2 h = \pi \times 1 \times 80$

= 251.3274122 feet^3

1 foot = 12 inches

So volume of tree = 251.3274122 × 12

= 3015.928464 inches^3

No. of matchsticks = 150796.44732

Is Sherida's solution realistic?

What do you like about Sherida's work?

What mistakes has Sherida made?

To help you to understand Sherida's work, what question(s) could you ask her?

How Did You Work?

In this questionnaire you are to compare your individual method to your group's method and your group's method to the Sample students' methods. Mark the boxes and complete the sentences that apply to your work.

1. Our group solution was mathematically better than my individual solution ☐ **OR** My individual solution was mathematically better than our group solution ☐

   This is because ---

   ---

2. We checked our method **Yes/No**

   We checked our method by --- We could check our method by ---

   --- ---

3. We made some assumptions **Yes/No**

   How did the assumptions affect your answer? ---

   ---

4. Our solution is similar to one of the sample responses ☐ **OR** Our solution is different from all the sample responses ☐

   Our solution is similar to ☐ *Add name of sample response*

   I prefer **our method** / **the sample response** method (*circle)* This is because ---

   This is because --- ---

   --- ---

   --- ---

Working Together

1. Share your method with your partner(s) and your ideas for improving your individual solution.

2. Together in your group, agree on the best method for completing
the problem.

3. Produce a poster, showing a joint solution to the problem.
   - Write down any assumptions you have made.

4. Check your work.

Make sure that everyone in the group can explain the reasons for your chosen method and describe any assumptions you have made.

Evaluating Sample Student Responses

1. Imagine you are the teacher and have to assess the student work.

2. Take turns to work through a student's solution.
   - Write your answers on your mini-whiteboards.

3. Explain your answer to the rest of the group.

4. Listen carefully to explanations.
   - Ask questions if you don't understand.

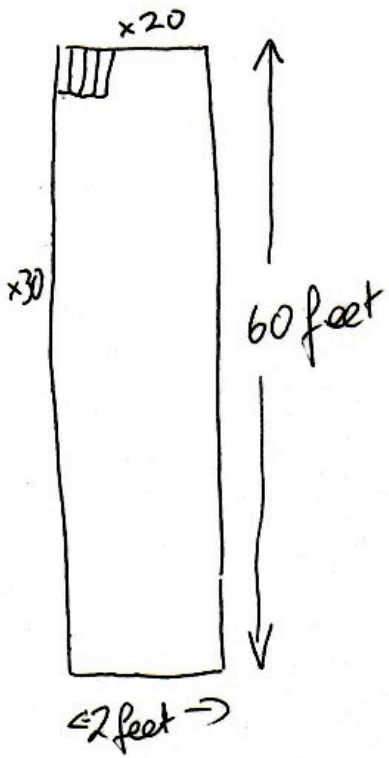
5. Once everyone is satisfied with the explanations, write the answers below the student's solution.
   - Make sure the student who writes the answers is not the student who explained them.

Sample Responses to Discuss: Jaabir

Height of tree = 80 feet

Narrows off at top so discount top 1/4

3/4 of 80 = 60



Matchstick width = $\frac{1}{10}$

So there will be 10 matches in 1 foot

– 20 matches in 2 feet

Matchstick length = 2.

So there will be 30 matches in 60 feet.

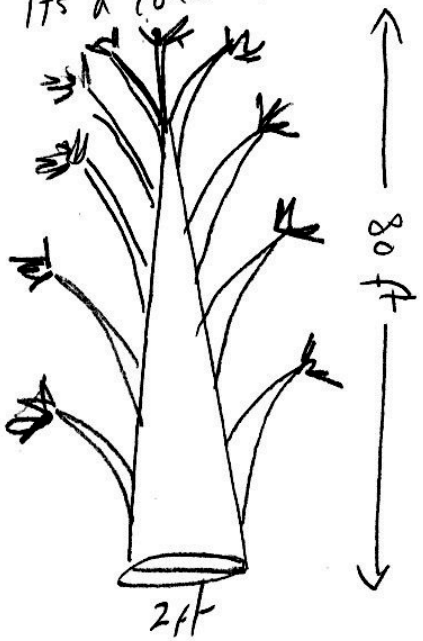
$20 \times 30 = 600$

Sample Responses to Discuss: Chan

Volume matchstick $\frac{1}{10} \times \frac{1}{10} \times 2 = 0.02$

Volume tree trunk ??

It's a cone so



$\frac{1}{3}\pi r^2 h$ is cone

$\frac{1}{3} \pi \times 2^2 \times 80$

$= 335.1032....$

$\approx 335.$ ~~1032...~~

0.02

$= 16755.2$ ~~matchsticks~~

$= 16750$ matchsticks

I didn't use the branches tho.

Sample Responses to Discuss: Sherida

Matchstick: $\frac{1}{10}$ inch by $\frac{1}{10}$ inch by 2 inches

Volume = 0.02 inches^3

Tree trunk is cylindrical so use volume of cylinder formula

Dimensions of tree: Height = 80 feet

Radius = $\frac{1}{2} \times 2$ feet = 1 foot

Volume of tree = $\pi r^2 h = \pi \times 1 \times 80$

= 251.3274122 feet^3

1 foot = 12 inches

So volume of tree = 251.3274122 $\times$ 12

= 3015.9289464 inches^3

No. of matchsticks = 150796.44732

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
Shell Center Team at the Center for Research in Mathematical Education
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org