CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

# Evaluating Statements about Radicals


Mathematics Assessment Resource Service

University of Nottingham & UC Berkeley

# Evaluating Statements about Radicals

## MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Use the properties of exponents, including rational exponents and manipulate algebraic statements involving radicals.
- Discriminate between equations and identities.

In this lesson there is also the (optional) opportunity to consider the role of the imaginary number $i = \sqrt{-1}$.

## STANDARDS ADDRESSED

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

N-RN: Extend the properties of exponents to rational exponents.
A-REI: Understand solving equations as a process of reasoning and explain the reasoning.
N-CN: Perform arithmetic operations with complex numbers. (optional)

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 3 and 6:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.

## INTRODUCTION

The unit is structured in the following way:

- Before the lesson, students work alone on an assessment task designed to reveal their current understanding. You review their work, creating questions to help students improve their solutions.
- During the lesson, students first work in small groups on a collaborative discussion task. After sharing their solutions with another group, students extend and generalize the math.
- An optional collaborative task focuses on imaginary numbers.
- During a whole-class discussion, students review the main mathematical concepts of the lesson.
- In a follow-up lesson, students review their initial solutions and then use what they have learned to revise the same introductory assessment task and complete a second, similar task.

## MATERIALS REQUIRED

- Each student will need a copy of *Operations with Radicals* and *Operations with Radicals (revisited)*, a mini-whiteboard, pen, and eraser.
- Each small group of students will need the *Card Set: Always, Sometimes, or Never True?*, scissors, glue, and a large sheet of poster paper. To begin with, cut *Cards G* and *H* off the sheet, but do not cut up the rest of the cards.
- There is a projector resource to support whole-class discussions. You may also want to copy the cards onto transparencies to be used on an overhead projector to support discussions.

## TIME NEEDED

15 minutes before the lesson, a 95-minute lesson (or two 50-minute lessons) (more if introducing imaginary numbers), and 20 minutes in a follow-up lesson. Exact timings will depend on the class.

## BEFORE THE LESSON

### Assessment task*: Operations with Radicals* (15 minutes)

Give this task in class or for homework a few days before the lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the next lesson.

Give each student a copy of the assessment task *Operations with Radicals*.

You may want to use the Slide P-1, *Operations with Radicals,* to display the instructions for this assessment task:

> *Read through the questions and try to answer them as carefully as you can.*
>
> *Throughout this task, the radical sign denotes a* ***positive*** *square root.*
>
> *We are only going to think about* ***positive or zero*** *values for x in all of these problems. So all the way through today's lesson,* ***x cannot be negative***.

**Operations with Radicals**

In the following statements:

- The symbol $\sqrt{\ }$ means the ***positive*** square root. (So $\sqrt{9} = +3$).
- Only consider values of $x$ ***greater than or equal to zero***.

For each statement, indicate whether it is true for all of these values of $x$, true for some values of $x$ or there are no values of $x$ for which it is true. Circle the correct answer.

If you choose 'Sometimes true', state all values of $x$ that make it true.

1. $\sqrt{\frac{x}{7}} = \frac{\sqrt{x}}{\sqrt{7}}$ Always true Sometimes true Never true

   It is true for ..............................

   Show your reasoning: ..............................

2. $\sqrt{x^2 + 3^2} = x + 3$ Always true Sometimes true Never true

   It is true for ..............................

   Show your reasoning: ..............................

3. $(1-\sqrt{2x})(1+\sqrt{2x}) = -5$ Always true Sometimes true Never true

   It is true for ..............................

   Show your reasoning: ..............................

It is important that, as far as possible, students are allowed to answer the questions without your help. Explain to students that they need not worry too much if they cannot understand nor do everything in the task. In the next lesson they will engage in similar work, which should help them to progress.

### Assessing students' responses

Collect students' responses to the task and think about what the students' work reveals about their current levels of understanding.

We suggest that you do not score students' work. The research shows that this will be counter-productive as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given in the *Common issues* table on the next page. We suggest that you make a list of your own questions, based on your students' work, using the ideas on the following page. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions, and highlight the questions relevant to each student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students. Note that the work should only be returned to students in the follow-up lesson.

| Common issues: | Suggested questions and prompts: |
|---|---|
| **Struggles to get started** | • Try some values to decide whether you think the equation can ever be true.<br>• How can you get rid of the radical sign(s) in the equation? |
| **Uses only guess and check** | • You have checked a few values of $x$.<br>How can you be sure there are no other values of $x$ for which this equation is true/false? |
| **Provides poor explanation for conclusions** | • How can you use math to explain your answer? |
| **Makes mistakes when manipulating the equations**<br>For example: The student equates expressions when they have only squared one side of the equation.<br>Or: The student subtracts a value from one side of the equation but not the other.<br>Or: The student assumes that squaring an expression in $x$ means that every $x$ in the expression just becomes $x^2$. | • You have changed the equation. How can you be sure that the two sides of the equation are equal? |
| **Does not choose appropriate operations when manipulating the equations**<br>For example: The student does not square to remove radical signs.<br>Or: The student does not simplify the expressions on each side. | • How could you manipulate the algebra to simplify the equation?<br>• What other operation could you use to change the equation? Why would that be a helpful move?<br>• What is the positive square root of an expression that is squared?<br>• Could squaring both sides of the equation help? |
| **Misinterprets the meaning of algebraic equations and expressions**<br>For example: The student assumes that because the expressions on each side of an equation are different, the equation can never be true. | • Try some values for $x$. Are there any values for $x$ that would make the equation true?<br>• Now solve the equation. |
| **Correctly answers the questions**<br>The student needs an extension task. | • Look at this inequality: $2\sqrt{x} > \sqrt{3x}$.<br>Is it always, sometimes, or never true? |

## SUGGESTED LESSON OUTLINE

### Whole-class introduction (15 minutes)

**During this lesson the radical sign, $\sqrt{\ }$ , denotes the positive square root. In all of the statements that we are going to look at in this lesson, we will consider values of *x greater than or equal to zero* only. Remember, *x cannot be negative* in these expressions.**

Give each student a calculator, mini-whiteboard, pen, and eraser.

Throughout the introduction, encourage students to first tackle a problem individually and only then discuss it with a neighbor. In that way students will have something to talk about. Maximize participation in the whole-class discussion by asking students to show you their solutions on their mini-whiteboards. Select a few students with interesting or contrasting answers to justify them to the class. Encourage the rest of the class to challenge these explanations.

You could say:

> *In this task, the radical sign denotes a **positive** square root. We are only going to think about **positive or zero** values for x in these problems, so throughout today's lesson, **x cannot be negative**.*

Then write this on the board:

| ***Always, Sometimes, or Never True?*** |
|---|
| $\sqrt{x-5} = \sqrt{x} - \sqrt{5}$ |

Then ask:

> *Is this statement always true, sometimes true, or never true?*

It is important that students understand the problem and spend some time thinking about what they need to include in their solutions. If students are struggling to get started, encourage them to choose a value to substitute into the equation.

Bring the class together and ask the following questions in turn:

> *Show me a value for x that makes the statement incorrect. Now show me another…*
>
> *Can you show me a value for x that makes the statement correct? [$x = 5$.]*
> *Can you show me another? [No.]*
> *How do you know for certain that there are no more?*
>
> *We can find many values for x that make the statement false and one value that makes it true. So is this statement: Always true, Sometimes true, or Never true? [Sometimes true.]*

You may find that students choose values of $x$ that are less than 5, thus making the left hand side of the equation imaginary. You may choose to address this issue in detail later in the lesson (*Collaborative activity 3*). For now, ask students to just use numbers that give non-negative values to the expression inside the square root. What is important is that they justify their classification, with reference to mathematically available criteria.

Students can see by inspection that the statement is sometimes, but not always true, but determining precisely the solution set will involve the use of algebra.

> *We know the statement is true for $x = 5$. How do we know for sure that this is the **only** value of x that makes the statement true? What can we do to figure this out?*

Allow students a few minutes to work on this problem. You may need to encourage some students to simplify the equation by finding a way to eliminate the radical signs. Then select two or three students to explain their work. If possible, select students who have made mistakes. A common one to look out for is that squaring a sum or difference of roots simply means removing the radical signs: $(\sqrt{x}-\sqrt{5})^2 = x-5$

Squaring the left hand side of the equation:

$$(\sqrt{x}-\sqrt{5})^2 = (\sqrt{x}-\sqrt{5})(\sqrt{x}-\sqrt{5}) = x-2\sqrt{x}\sqrt{5}+5$$

(Another common error here will be students putting – 5 at the end, instead of + 5.)

Squaring the right side:

$$\left(\sqrt{x\ -\ 5}\right)^2\ =\ x\ -\ 5$$

Students may be puzzled here why the – 5 remains as – 5 rather than + 5. It may help if they can see it as an instance of $\left(\sqrt{something}\right)^2 = something$.

The two expressions *may* be equal if:

$$x-5 = x-2\sqrt{x}\sqrt{5}+5$$
$$\Rightarrow 2\sqrt{x}\sqrt{5} = 10$$
$$\Rightarrow \sqrt{x}\sqrt{5} = 5$$
$$\Rightarrow x = 5$$

It is important that students appreciate that $x$ = 5 is the solution to the 'squared' equation $x-5=(\sqrt{x}-\sqrt{5})^2$ and *not necessarily* to the *original* equation $\sqrt{x-5} = \sqrt{x} - \sqrt{5}$ *before* squaring. Squaring both sides of an equation can introduce additional solutions. For example, if you begin with a simple statement such as $x$ = 3 and square both sides, you get the equation $x^2 = 9$, which has *two* solutions, 3 and –3. Squaring has introduced an extra solution. Likewise, if you start with the *false* statement –1 = 1, then squaring both sides gives the *true* statement 1 = 1, but this does not prove that the original statement was true! If you start, as here, by assuming something that is false, it is possible, by legitimate steps, to arrive at something that is true. (For example, 'All squares are triangles; all triangles are quadrilaterals; therefore, all squares are quadrilaterals'! The conclusion is true, even though the premises are not.) Students need to think carefully about what they are doing.

So it would normally be necessary to check that $x$ = 5 satisfies the original equation, but in fact in this case we had already done that at the start.

## Collaborative activity 1: classifying statements (20 minutes)

Organize the class into pairs. Give each pair a large sheet of paper, scissors, a glue stick, and the *Card Set: Always, Sometimes, or Never True?* with *Cards G* and *H* removed. (These cards could lead to a discussion about imaginary numbers and may be used later, if desired.) Explain to students how to divide their poster paper into three columns, headed *Always true*, *Sometimes true*, and *Never true*.

Explain the activity to students:

*You have a sheet of cards with statements on them.*
*Your task is to decide whether the statements are always, sometimes, or never true.*

*Have a look at the cards for a few minutes individually. Then, in your pairs, take turns to select and cut out a card that is interesting or challenging to you.*
*Explain your choice to your partner.*

*If you think the statement is sometimes true, you will need to find the values of $x$ for which it is true and the values of $x$ for which it is not true.*
*You will need to explain why the equation is true for just those values of $x$.*

*If you think the statement is always true, or never true, you will need to explain why you're sure of that. Remember, if you're showing it is always true or never true, substituting just a few values of $x$ is not enough.*

*Your partner then has to decide whether the card is in the correct column.*

*When you agree, glue down the card and your partner writes the explanations on the poster.*

You may want to display Slide P-2, *Always, Sometimes, or Never True?,* which summarizes the instructions for group work.

While students work in small groups, you have two tasks: to note students' approaches to the task and to use what you notice to support students' reasoning.

### Make a note of students' approaches to the task

Listen to and watch students carefully. In particular, be aware of whether or not students are addressing the difficulties they experienced in the assessment task.

- Which values do students choose to substitute into equations? Is their choice of values purposive, or does it seem random?
- Which operations do students choose to perform on the equations?
  Do they simplify expressions and manipulate the equations without errors?
  Do they make errors in the manipulation of exponents?
- What kinds of explanations do students give for their choices?
  Are they satisfied with a 'guess and check' approach? Or do they move to algebraic manipulation of the equations to find a general solution?
  Are they able to articulate the consequences of their algebraic work?
  Are partners convinced by the arguments?

You can use the information to support students' reasoning and also to focus the whole-class discussion at the end of the lesson.

### Support student reasoning

Try to avoid correcting students' errors for them. Instead, direct their attention to the error and ask them to explain what they have done. Use questions like those in the *Common issues* table to help students address their errors and misconceptions.

If students are having difficulties in moving toward an algebraic solution, use prompts to remind them of what they already know:

*Your equation is complicated in its current form. Is there any way to simplify it?*
*How will that help you find the values of $x$ for which the equation is true?*

*What do you know about manipulating algebraic expressions?*

*What do you know about operations on exponents? How might that be useful?*

Avoid making suggestions that move students towards a particular categorization of statements. Instead, prompt students to reason together.

Encourage your students to work algebraically, to develop convincing reasoning:

*You have shown the statement is true for this specific value of $x$.*
*Now convince me it is true for all values.*

*Can you use algebra to justify your decision for this card?*

You may need to explain the reasoning required for statements that are *sometimes* true. Some students will leave their responses at 'guess and check', having found a few values for which the equation is true and a few for which it is not true. In that case, emphasize that the task is to find exactly which values of $x$ make the equation true and which do not. The student also needs to explain why that result holds.

Once one student has explained the placement of a card, encourage their partner to participate either by explaining that reasoning again in his or her own words, or by challenging the reasons the first student gave.

*Alice, Gabriel placed that card in the 'Always true' column. Do you agree with her? OK, then explain it to me in your own words/why you don't agree.*

The purpose of this structured work is to encourage students to engage with each other's explanations and to take responsibility for each other's understanding.

It does not matter if students are not able to find a place for all the cards. It is more important that they understand the categorization of each card.

If you want the lesson to involve consideration of imaginary numbers, you may want to encourage learners to substitute some negative values also.

### Sharing posters (15 minutes)

As students finish placing the cards, ask one student from each group to make a quick copy on scrap paper of where cards were placed on his or her poster. This only involves writing the letters used to identify each card (A-F).

Ask one student from each group to visit another group's poster.

*If you are staying at your desk, be ready to explain the reasons for your group's placement of equations on your poster.*

*If you are visiting another group, take the scrap paper with you.*

*Go to another group's desk and look for differences.*
*Check to see which equations are in a different category from your own.*

*If there are differences, ask for an explanation.*
*If you still don't agree, explain your own thinking.*

*When you return to your desk, you need to consider whether to make changes to your own poster.*

You may want to display Slide P-3 *Sharing Posters*, which summarizes these instructions.

If a student finds that the group they visit has the cards in exactly the same places as they did in their group, the discussion can focus on whether their reasons were the same or different.

When students are satisfied that they have identified and discussed differences, they are to return to their own posters and negotiate with their partner any changes they think are needed. They can draw arrows on the poster to show the movement of a card and write an explanation next to the arrow.

### Extending the lesson over two days

If you are taking two days to complete the lesson unit then you may want to end the first lesson here. At the start of the second day, allow time for students to familiarize themselves with their posters before they have the opportunity to generalize their work in a whole-class discussion.

## Whole-class discussion: generalizing results (15 minutes)

This activity provides students with an opportunity to generalize their work in the previous task.

Write the first equation on the board again:

$$\sqrt{x-5} = \sqrt{x} - \sqrt{5}$$

*You decided before that the statement is sometimes true. It's true just when $x = 5$.*

*The equation has 'x's in it and '5's. Now change the '5's to another number.*
*Is the equation now always, sometimes or never true?*

Allow students a few minutes to think about this on their own and then give another couple of minutes to allow them to discuss their initial ideas in pairs.

Students may argue informally that changing the '5' to, for example a '4', will make the solution '4' instead. By doing this, they are using algebraic thinking, seeing the '5' as potentially a variable, such as $y$.

Write this statement on the board:

$$\sqrt{x-y} = \sqrt{x} - \sqrt{y}$$

*You can check for 5, 4, or any other number all at the same time if you can figure out when this algebraic equation is true. How might you do that?*

*Which operations did you use before, with $y = 5$?*

Ask students to work first individually and then in pairs, to show you how to manipulate the equation:

$$\sqrt{x-y} = \sqrt{x} - \sqrt{y}$$

Squaring the left side of the equation:

$$\left(\sqrt{x-y}\right)^2 = x - y$$

Squaring the right side of the equation:

$$\left(\sqrt{x} - \sqrt{y}\right)^2 = \left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} - \sqrt{y}\right)$$

$$= x + y - 2\sqrt{x}\sqrt{y}$$

The two expressions are equal when:

$$x - y = x + y - 2\sqrt{x}\sqrt{y}$$

$$2\sqrt{x}\sqrt{y} = 2y$$

$$\sqrt{x}\sqrt{y} = y$$

Squaring both sides:

$$xy = y^2$$

$$y\left(y - x\right) = 0$$

$y = 0$ and $y = x$.

A possible error is that students do not factorize and assume that the equation is true only when $x = y$.

Students must then substitute both of these possible solutions back into the original equation $\sqrt{x-y} = \sqrt{x} - \sqrt{y}$ in order to check that spurious additional solutions were not introduced in the two steps in which both sides of the equation were squared.

As they work on the problem, support the students as in the first collaborative activity. If necessary, discuss the solution as a class.

## Collaborative activity 2: generalizing results (15 minutes)

Once you have established how to replace the number with a variable:

*Choose one or two of the statements from your poster.*

*Replace the numbers in the equations with a variable, for example y.*

*Check to see if the statement should still remain in the same category.*

*For the statements you place in the 'Sometimes true' category, you should figure out exactly which values of x and y make the equation true.*

Students can write their equations and answers on the posters. Support the students as in the first collaborative activity.

## (Optional) Collaborative activity 3: introducing imaginary numbers (10 minutes)

If you want to work on imaginary numbers in the lesson, give each group *Cards G* and *H*. These two cards can be used to raise the issue of imaginary numbers.

*Now consider whether these new statements are always, sometimes, or never true.*

*What values of x make these statements true?*

Support student reasoning as before.

*What happens if x could be negative?*

*What is the result of substituting $x = -1$ into the statements?*

Students will find that for $x \geq 0$ there are no solutions for *Card G*, so it is 'never' true and just one solution, $x = 1$, for *Card H*, so that is 'sometimes' true. However, widening the domain of $x$ to include *all real numbers* leads to the additional solution $x = -1$ for both *Cards G* and *H*. Although this is a perfectly reasonable solution to the simplified equations, substituting it into the original equations *G* and *H* leads to square roots of negative numbers, which are imaginary.

## Whole-class discussion (15 minutes)

Ask students to display their posters at the front of the room for all to see.

Organize a whole-class discussion about the different strategies used to match the cards. Once one group has justified their choice for a particular placement, ask other students to contribute ideas of alternative approaches and their views on which reasoning method was easier to follow. The intention is that you focus on getting students to understand and share their **reasoning**, not just checking that everyone produced the right answers.

Ask each group to write on a mini-whiteboard an equation from their poster that meets some chosen criterion. For example, you might ask:

*Show me an equation that is 'Always true'.*
*Is your equation still always true when the equation contains two variables?*

*Show me an equation that is 'Sometimes true'.*
*For what values is it true for the equation with two variables?*

Encourage students to explain their answers using examples and justifications.

*Does anyone disagree with this classification? Why?*
*Which explanation do you prefer? Why?*

Draw out any issues you have noticed as students worked on the activity.

## (Optional) Extension: discussion of imaginary numbers using *Cards G* and *H*

If your students worked on *Cards G* and *H*, you may want to extend the whole-class discussion to include imaginary numbers.

Write the statement from *Card G* on the board.

$$\left(3 + \sqrt{x}\right)\left(3 - \sqrt{x}\right) = 10$$

*Is this statement ever true? If so, for which $x$ values is it true?*

Ask students to write the equation for *Card G* with $x = -1$ on their whiteboards.

*What do you notice when you substitute this value of x into the equation?*

*What do you think $\sqrt{-1}$ means?*

Ask students to put this imaginary number into context by thinking about their background knowledge of types of numbers.

*Tell me some of the different types of numbers you know.*

Push for a wide range of numbers, including rationals, decimals, negative numbers, irrational numbers such as $\sqrt{2}$, and transcendental numbers such as $3\pi$.

*These are all called 'real numbers'. The square root of a negative number is not a real number.*

*A new kind of number, which we call 'i', is used to define the imaginary number $\sqrt{-1}$.*

*You can use 'i' to represent lots of other imaginary numbers.*

*How can $\sqrt{-9}$ be represented as an imaginary number? [ $\sqrt{9 \times -1} = \sqrt{9} \times \sqrt{-1} = 3i$ .]*

*What about $\sqrt{-2}$ ? [ $\sqrt{2 \times -1} = \sqrt{2} \times \sqrt{-1} = \sqrt{2}i$ .]*

You could initiate a similar discussion related to *Card H*.

## Follow-up lesson: individual review (20 minutes)

Return the original *Operations with Radicals* assessment to the students, along with a new copy of the same task. If you have not added questions to individual pieces of work, then write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

Invite students to revise their work using your questions:

*I would like you to read through your original solutions and the questions I have written (on the board/on your script) to help you improve your work.*

*Answer these questions and revise your response. You may want to make some notes on your mini-whiteboard.*

Now give each student a copy of the *Operations with Radicals (revisited)* assessment task.

*Use what you have learned to answer these questions. Show all your work on the sheet and make sure you explain your answers really clearly.*

If students struggled with the original assessment task, you may feel it more appropriate for them to just spend their time revisiting *Operations with Radicals* rather than attempting *Operations with Radicals (revisited)* as well.

Some teachers give this as a homework task.

## SOLUTIONS

### Assessment task: *Operations with Radicals*

1. $\sqrt{\dfrac{x}{7}} = \dfrac{\sqrt{x}}{\sqrt{7}}$ This statement is **always true** when $x \geq 0$.

Some students might justify this claim by substituting a few values into $x$ and showing that the equation is true for those values. This is not sufficient justification. Others may just appeal directly to the laws for the manipulation of indices:

$$\left(\frac{a}{b}\right)^x = \frac{(a)^x}{(b)^x}.$$

for $b \neq 0$.

Whilst this shows that the statement is true, it does not help explain **why** the statement is always true. Students might attempt to use algebra:

Squaring the left side: $\left(\sqrt{\dfrac{x}{7}}\right)^2 = \dfrac{x}{7}$

Squaring the right side: $\left(\dfrac{\sqrt{x}}{\sqrt{7}}\right)^2 = \dfrac{x}{7}$.

So the statement is true for any values of $x$.

However, you could point out that this same approach of squaring both sides could be used to 'prove' that -1 = 1 is true!

2. $\sqrt{x^2 + 3^2} = x + 3$ The statement is **sometimes true**. It is true when $x = 0$.

Some students may use guess and check; others may provide a more general justification using algebra:

$\sqrt{x^2 + 3^2} = x + 3$

Squaring the left side: $x^2 + 9$

Squaring the right side: $(x + 3)^2 = x^2 + 6x + 9$

So the statement is true when: $x^2 + 9 = x^2 + 6x + 9$

$6x = 0; \ x = 0.$

Substituting back into $\sqrt{x^2 + 3^2} = x + 3$, we can see that $x = 0$ satisfies the original equation and so it must be the one and only solution.

3. $(1 - \sqrt{2x})(1 + \sqrt{2x}) = -5$ This statement is **sometimes true**. It is true when $x = 3$.

$\Rightarrow 1 - 2x = -5$

$\Rightarrow x = 3$

### Lesson task

For each card, students should identify values of $x$ for which the equation is true and explain why the equation is true (or not true) for just those values.

## Card A

$\sqrt{x+2} = \sqrt{x} + \sqrt{2}$ This equation is **sometimes true**. It is true when $x = 0$.

Squaring the right side:

$$\left(\sqrt{x} + \sqrt{2}\right)^2 = x + 2 + 2\sqrt{2}\sqrt{x}$$

Squaring the left side:

$$\left(\sqrt{x + 2}\right)^2 = x + 2$$

If the two sides are equal, then:

$$\begin{aligned} x + 2 &= x + 2 + 2\sqrt{2}\sqrt{x} \\ 2\sqrt{2}\sqrt{x} &= 0 \\ x &= 0. \end{aligned}$$

Checking, $x = 0$ satisfies the original equation.

When 2 is replaced with $y$, the equation is true if and only if $x = 0$ or $y = 0$. Logically this includes the possibility that both values are zero.

## Card B

$\sqrt{3x} = \sqrt{3} \times \sqrt{x}$ This statement is **always true** when $x \geq 0$.

Squaring the right hand side of the equation then taking the positive square root:

$$\sqrt{\left(\sqrt{3} \times \sqrt{x}\right)^2} = \sqrt{\left(\sqrt{3} \times \sqrt{x}\right) \times \left(\sqrt{3} \times \sqrt{x}\right)} = \sqrt{\sqrt{3} \times \sqrt{3} \times \sqrt{x} \times \sqrt{x}} = \sqrt{3 \times x} = \sqrt{3x}$$

Therefore this equation is an identity, true for any non-negative value of $x$. This holds true when 3 is replaced by $y$, provided that $y$ is also non-negative.

## Card C

$\sqrt{2 - x} = \sqrt{2} - \sqrt{x}$ This statement is **sometimes true**. It is true when $x = 2$ or when $x = 0$.

For a full solution, the student is asked not only to identify some values of $x$ for which the statement is true and some for which it is false, but to find *all* values of $x$ which make the statement true.

Squaring the right side:

$$\left(\sqrt{2} - \sqrt{x}\right)\left(\sqrt{2} - \sqrt{x}\right) = 2 + x - 2\sqrt{2}\sqrt{x}$$

Squaring the left side:

$$2 - x$$

If the two sides are equal, then:

$$\begin{aligned} 2 - x &= 2 + x - 2\sqrt{2}\sqrt{x} \\ 2x &= 2\sqrt{2}\sqrt{x} \\ x &= \sqrt{2}\sqrt{x} \end{aligned}$$

Squaring both sides:

$$\begin{aligned} x^2 &= 2x \\ x\left(x - 2\right) &= 0. \end{aligned}$$

$$x = 0 \quad \text{or} \quad x = 2.$$

These values satisfy the original equation.

When 2 is replaced with $y$, the equation is true when $x = y$ and $y = 0$.

## Card D

$$\sqrt{x^2 - 2^2} = x - 2$$

This equation is **sometimes true**. It is true when $x = 2$.

As with Card C, an error that students sometimes make is to assume that you can simply remove the squares and root notation from $\sqrt{x^2 - 2^2}$ and that the equation is thus always true.

Squaring the right side:

$$(x - 2)(x - 2) = x^2 + 4 - 4x$$

Squaring the left side:

$$x^2 - 2^2$$

If the two sides are equal, then:

$$x^2 - 2^2 = x^2 + 4 - 4x$$
$$4x = 8$$
$$x = 2.$$

This satisfies the original equation.

When 2 is replaced by $y$ then the equation is true when $x = y$ and when $y = 0$.

## Card E

$\sqrt{x^2} \times \sqrt{3^2} = 3x$ This equation is **always true** when $x \geq 0$.

When 3 is replaced by $y$ the equation is again **always true** provided that $y \geq 0$.

## Card F

$\dfrac{\sqrt{x^2}}{\sqrt{3^2}}$ This equation is **always true** when $x \geq 0$.

As with Card G, when 3 is replaced by $y$ the equation is again **always true** provided that $y > 0$. Note that this time we must exclude the possibility that $y=0$.

## Card G

$\left(3 + \sqrt{x}\right)\left(3 - \sqrt{x}\right) = 10$ This equation is **never true** within the domain of real numbers. The only solution is $x = -1$ and $x$ is not allowed to be negative.

Expanding the left-hand side,

$$9 - x = 10$$
$$x = -1.$$

Substituting this value back into the original equation:

$$\left(3 + \sqrt{-1}\right)\left(3 - \sqrt{-1}\right) = 10$$
$$(3 + i)(3 - i) = 10.$$

Thus, if students look only at values of x that are positive or zero, this statement is never true. However, if negative values of x are allowed and if we permit imaginary numbers, then there is a solution.

### Card H

$\sqrt{x+1} + \sqrt{x-1} = \sqrt{2x}$

This equation is **sometimes true**, when $x = 1$ and also, if imaginary numbers are allowed, when $x = -1$.

Squaring the left side:

$$\left(\sqrt{x+1} + \sqrt{x-1}\right)\left(\sqrt{x+1} + \sqrt{x-1}\right) = x + 1 + 2\sqrt{x+1}\sqrt{x-1} + x - 1$$

$$= 2\left(x + \sqrt{x+1}\sqrt{x-1}\right) = 2x + 2\sqrt{x^2 - 1}$$

Squaring the right side: $\left(\sqrt{2x}\right)^2 = 2x$

The two sides are equal if: $2x + 2\sqrt{x^2 - 1} = 2x$

$\Rightarrow \sqrt{x^2 - 1} = 0$

so $x^2 = 1$; $x = 1$ or $x = -1$.

In the latter case, both sides of the expression are imaginary numbers.

## Assessment task: *Operations with Radicals (revisited)*

These solutions are analogous to previous solutions, so all the reasoning is not repeated.

1. $\sqrt{x} \times \sqrt{x} = 2\sqrt{x}$

This statement is **sometimes true**. It is true when either $x = 0$ or $x = 4$.

Squaring both sides:

$x^2 = 4x$

$x(x-4) = 0$

$x = 0$ or $x = 4$.

If 2 were replaced by $y$ then the equation is sometimes true when $x = 0$ and when $x = y^2$

2. $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

This statement is **sometimes true**. It is true when $x \geq 0$ and $y > 0$. (Note that $x$ *can* be equal to zero, but $y$ *cannot*, since division by zero is undefined.)

Squaring the left side: $\left(\sqrt{\frac{x}{y}}\right)^2 = \frac{x}{y}$

Squaring the right side: $\left(\frac{\sqrt{x}}{\sqrt{y}}\right)^2 = \frac{x}{y}$.

*S*o the statement is true for any values of $x$ and $y$.

3. $\sqrt{3+x}+\sqrt{3-x}=2\sqrt{3}$ This statement is **sometimes true**. It is true when $x = 0$.

Square the left side:

$$\left(\sqrt{3+x}+\sqrt{3-x}\right)\left(\sqrt{3+x}+\sqrt{3-x}\right)=3+x+3-x+2\sqrt{3-x}\bullet\sqrt{3+x}$$

$$=6+2\sqrt{9-x^2}$$

Square the right side: 12

The sides are equal when:

$$6+2\sqrt{9-x^2}=12$$

$$2\sqrt{9-x^2}=6$$

$$\sqrt{9-x^2}=3$$

Square both sides

$$9-x^2=9$$

$$x^2=0$$

$$x=0$$

# Operations with Radicals

In the following statements:

- The symbol $\sqrt{\ }$ means the ***positive*** square root. (So $\sqrt{9} = +3$).
- Only consider values of $x$ ***greater than or equal to zero***.

For each statement, indicate whether it is true for all of these values of $x$, true for some values of $x$ or there are no values of $x$ for which it is true. Circle the correct answer.

If you choose 'Sometimes true', state all values of $x$ that make it true.

1. $\sqrt{\frac{x}{7}} = \frac{\sqrt{x}}{\sqrt{7}}$ Always true Sometimes true Never true

It is true for ..................................................

Show your reasoning: ..................................................

..................................................

..................................................

2. $\sqrt{x^2 + 3^2} = x + 3$ Always true Sometimes true Never true

It is true for ..................................................

Show your reasoning: ..................................................

..................................................

..................................................

3. $(1-\sqrt{2x})(1+\sqrt{2x}) = -5$ Always true Sometimes true Never true

It is true for ..................................................

Show your reasoning: ..................................................

..................................................

..................................................

# Card Set: Always, Sometimes, or Never True?

| | |
|---|---|
| A. $\sqrt{x+2}=\sqrt{x}+\sqrt{2}$ | B. $\sqrt{3x} = \sqrt{3} \bullet \sqrt{x}$ |
| C. $\sqrt{2-x} = \sqrt{2} - \sqrt{x}$ | D. $\sqrt{x^2 - 2^2} = x - 2$ |
| E. $\sqrt{x^2} \bullet \sqrt{3^2} = 3x$ | F. $\dfrac{\sqrt{x^2}}{\sqrt{3^2}} = \dfrac{x}{3}$ |
| G. $\left(3+\sqrt{x}\right)\left(3-\sqrt{x}\right)=10$ | H. $\sqrt{x+1}+\sqrt{x-1}=\sqrt{2x}$ |

# Operations with Radicals (revisited)

In the following statements:

- The symbol $\sqrt{\ }$ means the ***positive*** square root. (So $\sqrt{9} = +3$).
- Only consider values of $x$ ***greater than or equal to zero***.

For each statement, indicate whether it is true for all of these values of $x$, true for some values of $x$ or there are no values of $x$ for which it is true. Circle the correct answer.

If you choose 'Sometimes true', state all values of $x$ that make it true.

1. $\sqrt{x} \times \sqrt{x} = 2\sqrt{x}$ Always true Sometimes true Never true

It is true for ..........................................................................

Show your reasoning: ..........................................................................

..........................................................................

..........................................................................

If you replace 2 by $y$, is your answer still correct? Explain.

..........................................................................

2. $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$ Always true Sometimes true Never true

It is true for ..........................................................................

Show your reasoning: ..........................................................................

..........................................................................

..........................................................................

3. $\sqrt{3 + x} + \sqrt{3 - x} = 2\sqrt{3}$ Always true Sometimes true Never true

It is true for ..........................................................................

Show your reasoning: ..........................................................................

..........................................................................

..........................................................................

# Operations with Radicals

- Read through the questions and try to answer them as carefully as you can.
- Throughout this task, the radical sign denotes a ***positive*** square root.
- We are only going to think about ***positive or zero*** values for $x$ in all of these problems. So all the way through today's lesson, $x$ ***cannot be negative***.

# Always, Sometimes, or Never True?

1. Take turns to select and cut out a card that interests or challenges you.
2. If you think the equation is '**Sometimes true**':
   - Find values of $x$ for which the equation is true and values of $x$ for which it is not true.
   - Explain why the equation is true for just those values of $x$.

   If you think the equation is '**Always true**' or '**Never true**':
   - Explain how you can be sure that this is the case.
3. Your partner then decides if the card is in the correct column.
   - If you think it is correctly placed, explain why.
   - If you disagree, move the card and explain your reasoning.
4. Once you agree, glue down the card and your partner writes an explanation on the poster next to the card.

# Sharing Posters

1. If you are staying at your desk, be ready to explain the reasons for the placement of cards on your poster.

2. If you are visiting another group:
   - Copy the table from your poster onto a piece of paper.
   - Go to another group and read their poster.
   - Check: Do they place any cards in categories that differ from your group’s poster?
   - If there are differences, ask for an explanation.
   - If you still don't agree, explain your own thinking.

3. Return to your own group.
   - Do you need to make any changes to your own poster?

Mathematics Assessment Project

# Classroom Challenges

These materials were designed and developed by the
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