CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Sorting Equations and Identities

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Sorting Equations and Identities

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Recognize the differences between equations and identities.
- Substitute numbers into algebraic statements in order to test their validity in special cases.
- Resist common errors when manipulating expressions such as $2(x - 3) = 2x - 3$; $(x + 3)^2 = x^2 + 3^2$.
- Carry out correct algebraic manipulations.

It also aims to encourage discussion on some common misconceptions about algebra.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

A-REI: Solve equations and inequalities in one variable.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 2, 3, 6, and 7:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

The lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task that is designed to reveal their current understanding and difficulties. You then review their work and create questions for students to answer in order to improve their solutions.
- After a whole-class introduction, students work in small groups on a collaborative discussion task.
- Students return to their original task and try to improve their own responses.

MATERIALS REQUIRED

- Each student will need two copies of the assessment task *Equations and Identities,* a mini-whiteboard, a pen, and an eraser.
- Each small group of students will need *Card Set*: *Always, Sometimes, or Never True?* (cut into cards before the lesson), a marker pen, a glue stick, and a large sheet of paper for making a poster.
- There is a projector resource to support the whole-class introduction.

TIME NEEDED

15 minutes before the lesson, a 1-hour lesson, and 15 minutes in a follow-up lesson. Timings given are approximate and will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Equations and Identities* (15 minutes)

Give this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.

Give each student a copy of the *Equations and Identities* task.

> *Read through the questions and try to answer them as carefully as you can.*

It is important that, as far as possible, students are allowed to answer the questions without your assistance.

Students should not worry too much if they cannot understand or do everything, because in the next lesson they will work on a similar task, which should help them. Explain to students that by the end of the next lesson, they should be able to answer questions such as these confidently. This is their goal.

Equations and Identities

1. Write down an example of an equation that has:

   (a) One solution. ....

   (b) Two solutions. ....

   (c) An infinite number of solutions. ....

   (d) No solutions. ....

2. For each of the following statements, indicate whether it is "Always true," "Never true," or "Sometimes true." Circle the correct answer. If you choose "Sometimes true" then state on the line below when it is true. The first one is done for you as an example.

$x + 2 = 3$	Always true It is true when $\boldsymbol{x = 1}$**.**	Never true	Sometimes true
$x - 12 = x + 30$	Always true It is true when	Never true	Sometimes true
$2(x + 6) = 2x + 12$	Always true It is true when	Never true	Sometimes true
$3(x - 2) = 3x - 2$	Always true It is true when	Never true	Sometimes true
$(x + 4)^2 = x^2 + 4^2$	Always true It is true when	Never true	Sometimes true
$x^2 + 4 = 0$	Always true It is true when	Never true	Sometimes true

3. Which of the equations in question 2 are also identities?

   ....

   ....

   In your own words, explain what is meant by an identity.

   ....

Assessing students' responses

Collect students' responses to the task and make some notes on what their work reveals about their current levels of understanding. The purpose of doing this is to forewarn you of the difficulties students will experience during the lesson itself, so that you may prepare carefully.

We suggest that you do not write scores on students' work. The research shows that this is counterproductive as it encourages students to compare scores and distracts their attention from what they are to do to improve their mathematics.

Instead, help students to make further progress by asking questions that focus their attention on aspects of their work. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students in the follow-up lesson.

Common issues:	**Suggested questions and prompts:**
Writes expressions rather than equations For example: The student writes $y + 3$ for an equation with an infinite number of solutions.	• What is the difference between an equation and an expression? • How can you change your expression into an equation?
Fails to include a variable in their equation For example: The student has written $5 + 5 = 10$ as an example of an equation with one solution.	• Can you include an unknown number or a variable in the equation so that we can look at all possible values of that unknown?
Fails to provide an example of an equation with an infinite number of solutions	• What would an equation with an infinite number of solutions look like?
Provides a quadratic with non-integer solutions as an example of an equation with no solutions For example: The student gives $x^2 + 8x + 13 = 0$ as an answer to Q1d. The student has assumed that because it won't factorize there are no solutions.	• Can a quadratic equation that will not factorize still have solutions/cross the x-axis? How can you check whether or not a quadratic equation has solutions?
Assumes that $-(x^2)$ is the same as $(-x)^2$ For example: The student classifies $x^2 + 4 = 0$ as true when $x = -2$.	• What does $(-x)^2$ mean? What kind of number do we get when we multiply two negative numbers together? • Is x^2 positive or negative?
Correctly answers all the questions The student needs an extension task.	• Use algebra to justify one of your answers to Question 2. • Draw a diagram to justify one of your answers to Question 2.

SUGGESTED LESSON OUTLINE

Whole-class introduction (15 minutes)

Give each student a mini-whiteboard, a pen, and an eraser. Write the following equation on the board:

$$(x + 2)(y + 2) = xy + 4$$

Is this equation 'always true', 'never true' or 'sometimes true'? [Write 'always', 'never' or 'sometimes' on your whiteboard.]

Typically, most students will begin by saying that this is never true.

Can you show me values for x and y that make the equation false?

Can you show me values for x and y that make the equation true?

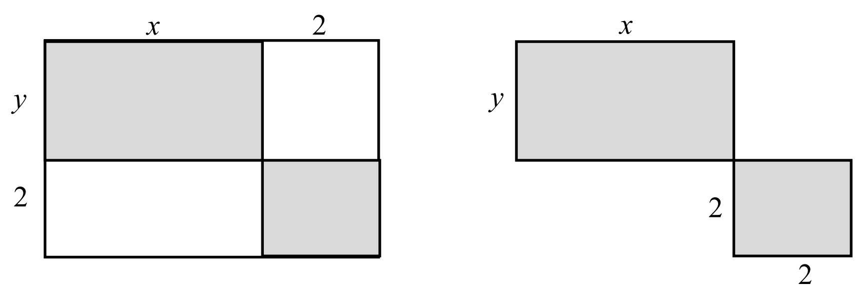
Hold a discussion about the responses, asking students to provide values for x and y to support their response.

Can the values of x and y be the same number? Can you figure out one?

This misconception needs to be explicitly addressed. Some students may assume that because x and y are different letters, they should take different values.

Students may spot that the equation is true when $x = y = 0$.

If students are struggling to find any values of x and y for which the equation is true, drawing an area diagram may be helpful (Slide P-1):



Total area = $(x + 2)(y + 2)$ Total area = $xy + 4$

For these two area diagrams to be equal, what are the values of x and y?

For the area diagrams to be the equal, $2y$ must equal 0 and $2x$ must equal 0. This is true when x and y are both equal to 0.

When students are comfortable that when $x = y = 0$ the equation is true, ask them to summarize their findings.

We have found values of x and y that make the equation false and values of x and y that make the equation true. Is the equation always, sometimes or never true? [Equation is sometimes true.]

Next, ask the students:

Are x = 0 and y = 0, the only values that make the equation true? How could we find out?

Using an algebraic approach here might be helpful, as we are unable to describe a negative area. The following method may be appropriate:

$$(x + 2)(y + 2) = xy + 2x + 2y + 4.$$

We want to know when this is the same as $xy + 4$, which must be when $2x + 2y = 0$, i.e. when $x + y = 0$ or when $x = -y$. We can therefore conclude that the equation is true when $x = -y$.

Now write this equation on the board:

$$(x + 2)(x - 2) = x^2 + 4$$

How about this equation? Is it 'always true', 'never true' or 'sometimes true'?

Students will probably find values for x for which the equation is false. After a discussion of a couple of these examples, encourage students to justify their conclusions:

Give me a value of x that will make the equation false/true? And another? [There are no values of x that will make the equation true.]

Do you think the equation is never true? Convince me. [Students should simplify the left side of the equation to $x^2 - 4$. The equation is never true, because $4 \neq -4$.]

After a few minutes, ask one or two students to explain their answers. Encourage other students to challenge their reasoning.

In this activity, the students use the term **identity**.

*If an equation is always true, we say it is an **identity**.*

Teachers may be accustomed to varying uses of the term 'identity'. While this is not the main focus of this activity, for the purpose of the lesson, the term 'identity' is used to describe equations that are always true.

Collaborative activity: *Always, Sometimes, or Never True?* (25 minutes)

Ask students to work in groups of two or three.

Give each group *Card Set*: *Always, Sometimes, or Never True?*, a large sheet of paper, a marker pen, and a glue stick.

Ask students to divide their large sheet into three columns and head respective columns with the words: *Always True*, *Sometimes True*, *Never True*.

You may want to use Slide P-2 of the projector resource to display the following instructions:

You are now going to consider whether the equations on your desk are 'Always', 'Sometimes', or 'Never True'.

In your groups, take turns to place a card in a column and justify your answer to your partner.

If you think the equation is sometimes true, you will need to find values of x for which it is true and values of x for which it is not true.

If you think the equation is always true or never true, you will need to explain how we can be sure that this is the case. Remember, showing it is true, or never true, for just a few values is not sufficient.

Another member of the group should then either explain their reasoning again in his or her own words, or challenge the reasons you gave.

It is important that everyone in the group understands the categorization of each card.

When everyone in the group agrees, glue the card onto your poster. Write the reason for your choice of category next to the card.

It does not matter if you do not manage to place all of the cards. It is more important that everyone in the group understands the categorization of each card.

The purpose of this structured work is to encourage each student to engage with their partner's explanations and to take responsibility for their partner's understanding.

While students work in small groups you have two tasks: to make a note of student approaches to the task and to support student reasoning.

Make a note of student approaches to the task

Listen and watch students carefully. In particular, listen to see whether students are addressing the difficulties they experienced in the assessment. You can use this information to focus the whole-class discussion towards the end of the lesson.

Support student reasoning

Use the questions in the *Common issues* table to help address misconceptions.

Encourage students to explain their reasoning carefully.

> *You have shown the statement is true for this specific value of x. Now convince me it is always true for every number!*
>
> *Can you use algebra to justify your decision for this card?*
>
> *Can you draw a diagram to explain your categorization for this card?*
>
> *(Card 8) Can you sketch a graph to show why $x^2 = 2x$ has only two solutions?*
>
> *(Card 9) Draw an area diagram to show that $(x + 3)^2$ means something different from $x^2 + 3^2$.*
>
> *(Card 11) Can you draw an area diagram to show why $(3x)^2$ is always equal to $9x^2$?*

If some students try to solve the equations by algebraic manipulation, they may notice that while sometimes this gives them possible solutions, sometimes they just get $0 = 0$. These are, of course, the identities. Equations that have no solutions give absurdities such as $1 = 2$.

If students finish the task quickly, ask them to create new examples.

> *Can you make up an identity? And another one?*
>
> *Can you make up an equation that has two solutions?*
>
> *Can you make up an equation that has no solutions and shows a common algebraic mistake? [E.g. $3(x + 4) = 3x + 4$.]*

Whole-class discussion (20 minutes)

Organize a whole-class discussion about different methods of justification used for two or three equations.

Ask each group to choose an equation from their poster that meets some given criteria. For example:

> *Show me an equation that has no solutions.*
>
> *Show me an equation that has just one solution. Write this solution on your mini-whiteboard.*
>
> *Show me an equation that has two solutions. What are they?*
>
> *Show me an equation that has an infinite number of solutions.*
>
> *Show me an identity.*

You may find that numerous different equations are displayed in response to a given criterion. If more than one group shows the same equation, ask each of these groups of students to give a justification of their thinking. Then ask other students to contribute ideas of alternative approaches and their views on which reasoning method was easier to follow. It is important that students consider a variety of methods and begin to develop a repertoire of approaches.

Why did you put this equation in this column? How else can you explain that decision?

Can anyone improve this explanation?

Which explanation do you prefer? Why?

Draw out issues you have noticed as students worked on the activity. Make specific reference to the misconceptions you noticed during the collaborative activity.

Follow-up lesson: improving individual solutions to the assessment task (15 minutes)

Return their original assessment *Equations and Identities* to the students, together with a second blank copy of the task.

If you have not added questions to individual pieces of work, write your list of questions on the board. Students should select from this list only the questions they think are appropriate to their own work.

Look at your original response and the questions (on the board/written on your script.)

Answer these questions and using what you have learned, revise your work.

Some teachers give this for homework.

SOLUTIONS

Always true (Identities)	Sometimes true	Never true
$2(x+3)=2x+6$	$x-6=6-x$ True when $x=6$.	$2(x-3)=2x-3$
$x^2-1=(x+1)(x-1)$	$x+6=y+6$ True when $x=y$.	$x^2+6=0$ (unless you include complex numbers.)
$(x-6)^2=(6-x)^2$	$\frac{x}{6}=\frac{6}{x}$ True when x = +6 or −6.	
$(3x)^2=9x^2$	$6+2x=8x$ True when x = 1.	
	$x^2=2x$ True when x = 0 or 2.	
	$(x+3)^2=x^2+3^2$ True when x = 0.	
	$(x+1)(x+4)=x^2+14$ True when x = 2.	
	$\frac{x+6}{2}=x+3$ True when x = 0.	

Equations and Identities

1. Write down an example of an equation that has:

   (a) One solution. ____________________

   (b) Two solutions. ____________________

   (c) An infinite number of solutions. ____________________

   (d) No solutions. ____________________

2. For each of the following statements, indicate whether it is 'Always true', 'Never true' or 'Sometimes true'. Circle the correct answer. If you choose 'Sometimes true' then state on the line below when it is true. The first one is done for you as an example.

$x + 2 = 3$	Always true	Never true	Sometimes true
	It is true when $x = 1$.		
$x - 12 = x + 30$	Always true	Never true	Sometimes true
	It is true when ____________________		
$2(x + 6) = 2x + 12$	Always true	Never true	Sometimes true
	It is true when ____________________		
$3(x - 2) = 3x - 2$	Always true	Never true	Sometimes true
	It is true when ____________________		
$(x + 4)^2 = x^2 + 4^2$	Always true	Never true	Sometimes true
	It is true when ____________________		
$x^2 + 4 = 0$	Always true	Never true	Sometimes true
	It is true when ____________________		

3. Which of the equations in question 2 are also identities?

   ____

   ____

   In your own words, explain what is meant by an identity.

   ____

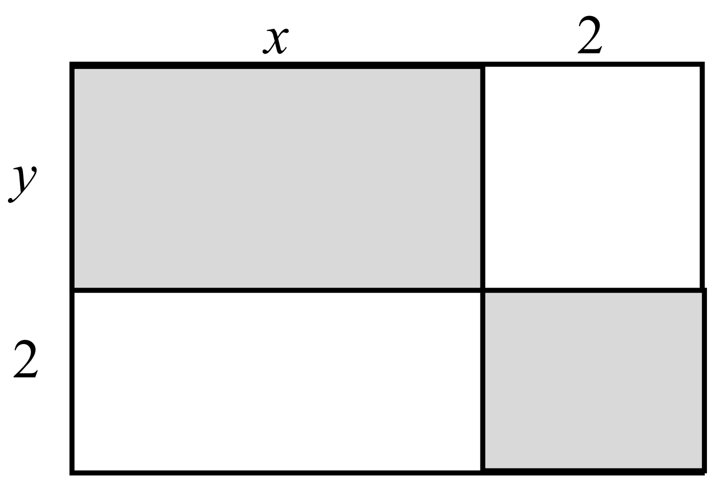
   ____

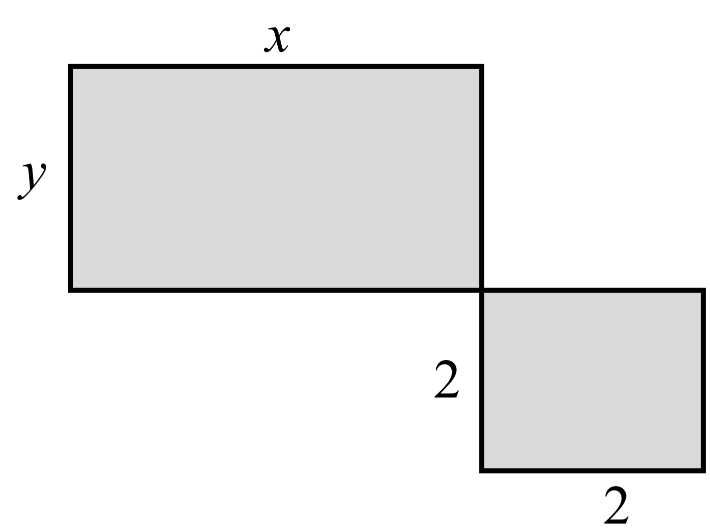
Card Set: Always, Sometimes, or Never True?

1 $$x-6=6-x$$	2 $$x+6=y+6$$
3 $$\frac{x}{6}=\frac{6}{x}$$	4 $$6+2x=8x$$
5 $$2(x-3)=2x-3$$	6 $$2(x+3)=2x+6$$
7 $$\frac{x+6}{2}=x+3$$	8 $$x^2=2x$$
9 $$(x+3)^2=x^2+3^2$$	10 $$(x-6)^2=(6-x)^2$$
11 $$(3x)^2=9x^2$$	12 $$x^2-1=(x+1)(x-1)$$
13 $$x^2+6=0$$	14 $$(x+1)(x+4)=x^2+14$$

Always, Sometimes, or Never True?

$$(x + 2)(y + 2) = xy + 4$$





Always, Sometimes, or Never True?

- In your groups, take turns to place a card in a column and justify your answer to your partner.
- If you think the equation is ‘sometimes true’, find values of x for which it is true and values of x for which it is not true.
- If you think the equation is ‘always true’ or ‘never true’, explain how we can be sure that this is the case.
- Another member of the group should then either explain that reasoning again in his or her own words, or challenge the reasons you gave.
- When everyone in the group agrees, glue the card onto the poster. Write the reason for your choice next to the card.

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org