CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Interpreting Algebraic Expressions

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Interpreting Algebraic Expressions

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to translate between words, symbols, tables, and area representations of algebraic expressions. It will help you to identify and support students who have difficulty:

- Recognizing the order of algebraic operations.
- Recognizing equivalent expressions.
- Understanding the distributive laws of multiplication and division over addition (expansion of parentheses).

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

A-SSE: Interpret the structure of expressions.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 1, 2, and 7:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

The lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task designed to reveal their current understanding and difficulties. You then review their work and formulate questions for students to answer, to help them improve their solutions.
- During the lesson, students work in pairs or threes to translate between word, symbol, table of values, and area representations of expressions.
- In a whole-class discussion, students find different representations of expressions and explain their answers.
- Finally, students return to their original assessment task and try to improve their own responses.

MATERIALS REQUIRED

- Each student will need two copies of *Interpreting Expressions*, a mini-whiteboard, pen, and eraser.
- Each pair of students will need glue, a felt-tipped pen, a large sheet of poster paper, and cut-up copies of *Card Set A: Expressions, Card Set B: Words, Card Set C: Tables,* and *Card Set D: Areas*. Note that the blank cards are part of the activity.
- If you think you will need to continue with the activities into a second lesson, provide envelopes and paper clips for storing matched cards between lessons.

TIME NEEDED

10 minutes for the assessment task, a 90-minute lesson (or two 50-minute lessons), and 10 minutes in a follow-up lesson. All timings are approximate and will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Interpreting Expressions* (10 minutes)

Have students do this task in class or for homework a day or more before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.

Give each student a copy of *Interpreting Expressions*.

I want you to spend ten minutes working individually on this task.

Don't worry too much if you can't understand or do everything. There will be a lesson [tomorrow] with a similar task that will help you improve.

It is important that, as far as possible, students are allowed to answer the questions without assistance.

If students are struggling to get started, ask them questions that help them to understand what is required, but do not do the task for them.

Interpreting Expressions

1. Write algebraic expressions for each of the following:

   a. Multiply ***n*** by 5 then add 4. ....

   b. Add 4 to ***n*** then multiply by 5. ....

   c. Add 4 to ***n*** then divide by 5. ....

   d. Multiply ***n*** by ***n*** then multiply by 3. ....

   e. Multiply ***n*** by 3 then square the result. ....

2. The equations below were created by students who were asked to write equivalent expressions on either side of the equals sign.

   Imagine you are a teacher. Your job is to decide whether their work is right or wrong. If you see an equation that is false, then:

   a. Cross out the expression on the right and replace it with an expression that is equivalent to the one on the left.

   b. Explain what is wrong, using words or diagrams.

$$2(n + 3) = 2n + 3$$

$$\frac{10n - 5}{5} = 2n - 1$$

$$(5n)^2 = 5n^2$$

$$(n + 3)^2 = n^2 + 3^2 = n^2 + 9$$

Assessing students' responses

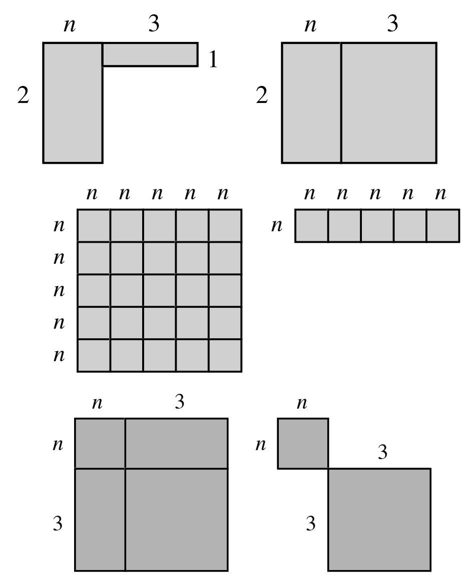
Collect students' responses to the task. Make some notes about what their work reveals about their current levels of understanding. The purpose of doing this is to forewarn you of the difficulties students will experience during the lesson itself, so that you may prepare carefully.

We suggest that you do not score students' papers. The research shows that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given in the *Common issues* table on the next page. We suggest that you make a list of your own questions, based on your students' work, using the ideas on the following page. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions relevant to each student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students in the follow-up lesson.

Common issues:	Suggested questions and prompts:
Writes expressions left to right, showing little understanding of the order of operations implied by the symbolic representation For example: The student writes: Q1a. $n \times 5 + 4$ (not incorrect). Q1b. $4 + n \times 5$. Q1c. $4 + n \div 5$. Q1d. $n \times n \times 3$.	• Can you write answers to the following? $4 + 1 \times 5$ $4 + 2 \times 5$ $4 + 3 \times 5$ • Check your answers with your (scientific) calculator. How is your calculator working these out? • So what does $4 + n \times 5$ mean? Is this the same as Q1b?
Does not construct parentheses correctly or expands them incorrectly For example: The student writes: Q1b. $4 + n \times 5$ instead of $5(n + 4)$. Q1c. $4 + n \div 5$ instead of $\frac{4+n}{5}$. Or: The student counts: Q2. $2(n+3) = 2n+3$ as correct. Q2. $(5n)^2 = 5n^2$ as correct. Q2. $(n+3)^2 = n^2 + 3^2$ as correct.	• Which one of the following is the odd one out: *Think of a number, add 3, and then multiply your answer by 2.* *Think of a number, multiply it by 2, and then add 3.* *Think of a number, multiply it by 2, and then add 6.* Why?
Identifies errors but does not give explanations For example: The student corrects the first, third, and fourth statements, but no explanation or diagram is used to explain why they are incorrect (Q2).	• How would you write expressions for these areas?  • Can you do this in different ways?

SUGGESTED LESSON OUTLINE

Interactive whole-class introduction (10 minutes)

Give each student a mini-whiteboard, pen, and eraser and hold a short question and answer session. If students show any incorrect answers, write the correct answer on the board and discuss any problems.

On your mini-whiteboards, show me an algebraic expression that means:

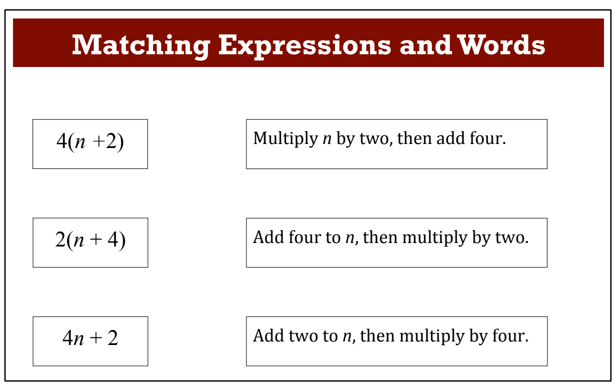
Multiply n by 4 and then add 3 to your answer.	$4n+3$
Add 3 to n and then multiply your answer by 4.	$4(3+n)$
Add 5 to n and then divide your answer by 3.	$\frac{n+5}{3}$
Multiply n by n and then multiply your answer by 5.	$5n^2$
Multiply n by 5 and then square your answer.	$(5n)^2$

Collaborative activity 1: matching expressions and words (20 minutes)

The first activity is designed to help students interpret symbols and realize that the way the symbols are written defines the order of operations.

Organize students into groups of two or three.

Display Slide P-1 of the projector resource:



Note that one of the algebraic expressions does not have a match in words. This is deliberate! It is to help you explain the task to students.

Model the activity briefly for students, using the examples on the projector resource:

I am going to give each group two sets of cards, one with expressions written in algebra and the other with words.

Take turns to choose an expression and find the words that match it. [4(n + 2) matches 'Add 2 to n then multiply by 4'; 2(n + 4) matches 'Add 4 to n then multiply by 2'.]

When you are working in groups, you should place these cards side by side on the table and explain how you know that they match.

If you cannot find a matching card, then you should write your own using the blank cards provided. [4n + 2 does not match any of the word cards shown on Slide P-1. The word card 'Multiply n by two, then add four' does not match any of the expressions.]

Give each small group of students a cut-up copy of *Card Set A: Expressions* and *Card Set B: Words*:

Card Set A: Expressions	
E1 $\frac{n+6}{2}$	E2 $3n^2$
E3 $2n+12$	E4 $2n+6$
E5 $2(n+3)$	E6 $\frac{n}{2}+6$
E7 $(3n)^2$	E8 $(n+6)^2$
E9 $n^2+12n+36$	E10 $3+\frac{n}{2}$
E11 n^2+6	E12 n^2+6^2
E13	E14

Card Set B: Words	
W1 Multiply ***n*** by two, then add six.	W2 Multiply ***n*** by three, then square the answer.
W3 Add six to ***n*** then multiply by two.	W4 Add six to ***n*** then divide by two.
W5 Add three to ***n*** then multiply by two.	W6 Add six to ***n*** then square the answer.
W7 Multiply ***n*** by two then add twelve.	W8 Divide ***n*** by two then add six.
W9 Square ***n***, then add six	W10 Square ***n***, then multiply by nine
W11	W12
W13	W14

Support students in making matches and explaining their decisions. As they do this, encourage them to speak the algebraic expressions out loud. Students may not be used to 'talking algebra' and may not know how to say what is written, or may do so in ways that create ambiguities.

For example, the following conversation between a teacher and student is fairly typical:

Teacher: *Tell me in words what this one says. [Teacher writes:* $3+\frac{n}{2}$ *.]*

Student: *Three add n divided by two.*

Teacher: *How would you read this one then? [Teacher writes:* $\frac{(3+n)}{2}$ *.]*

Student: *Three add n divided by two. Oh, but in the second one you are dividing it all by two.*

Teacher: *So can you try reading the first one again, so it sounds different from the second one?*

Student: *Three add … [pause] … n divided by two [said quickly]. Or n divided by two, then add three.*

Students will need to make word cards to match E10: $3+\frac{n}{2}$ and E12: n^2+6^2.

They will also need to make expression cards to match W3: *Add 6 to n, then multiply by 2* and W10: *Square n, then multiply by 9*.

Some students may notice that some expressions are equivalent, for example $2(n + 3)$ and $2n + 6$. You do not need to comment on this now as when *Card Set C: Tables* is given out, students will be able to notice this for themselves.

Collaborative activity 2: matching expressions, words, and tables (20 minutes)

Give each small group of students a cut-up copy of *Card Set C: Tables*:

Card Set C: Tables

T1

n	1	2	3	4
Ans	14	16	18	20

T2

n	1	2	3	4
Ans			81	144

T3

n	1	2	3	4
Ans		10	15	22

T4

n	1	2	3	4
Ans	3		27	48

T5

n	1	2	3	4
Ans			81	100

T6

n	1	2	3	4
Ans		10	12	14

T7

n	1	2	3	4
Ans		4		5

T8

n	1	2	3	4
Ans	6.5	7	7.5	8

Card Set C: Tables will make students substitute numbers into the expressions and will alert them to the fact that different expressions are equivalent.

Ask students to match these new cards to the two card sets they have been working on. Some tables have numbers missing and students will need to write these in.

Encourage students to use strategies for matching. There are shortcuts that will help to minimize the work. For example, some may notice that:

Since $2(n + 3)$ is an even number, we can just look at tables with even numbers in them.

Since $(3n)^2$ is a square number, we can look for tables with only square numbers in them.

Students will notice that there are fewer tables than expressions. This is because some tables match more than one expression. Allow students time to discover this for themselves. As they do so, encourage them to test that they match for all *n*. This is the beginning of a generalization.

Do $2(n + 3)$ and $2n + 6$ always give the same answer when $n = 1, 2, 3, 4, 5$?

What about when $n = 3246$, or when $n = -23$, or when $n = 0.245$?

Check on your calculator.

Can you explain how you can be sure?

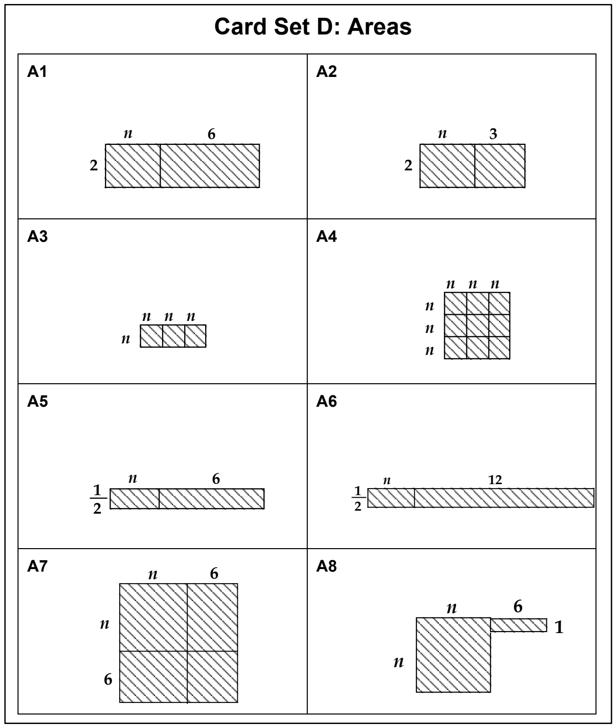
This last question is an important one and will be followed up in the next part of the lesson.

Extending the lesson over two days

It is important not to rush the learning. At about this point, some lessons run out of time. If this happens, ask pupils to stack their cards in order, so that matching cards are grouped together and fasten them with a paper clip. Ask students to write their names on an envelope and store the matched cards in it. These envelopes can then be reissued at the start of next lesson.

Collaborative activity 3: matching expressions, words, tables, and areas (20 minutes)

Give each small group of students a cut-up copy of the *Card Set D: Areas*, a large sheet of paper, a felt-tipped pen, and a glue stick.



The *Card Set D: Areas* will help students to understand *why* the different expressions match the same tables of numbers.

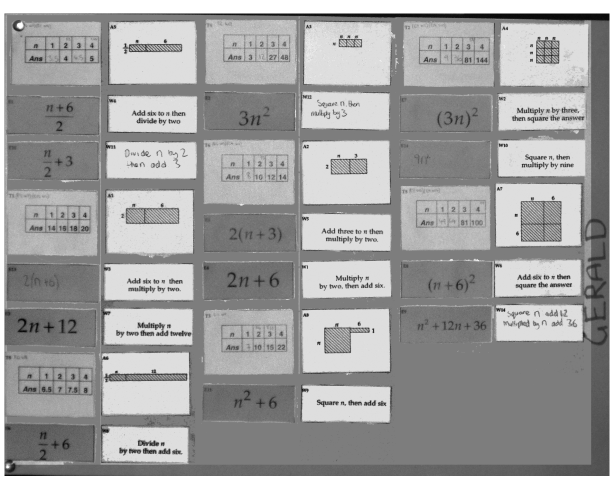
Each of these cards shows an area.

I want you to match these area cards to the cards already on the table.

When you reach agreement, paste down your final arrangement of cards onto the large sheet of paper, creating a poster.

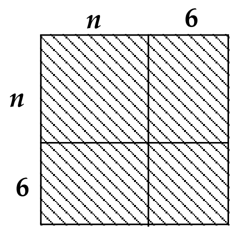
Next to each group of cards write down why the areas show that different expressions are equivalent.

The posters students produce will need to be displayed in the final whole-class discussion. They may look something like this:



As students match the cards, encourage them to explain and write down **why** particular pairs of cards go together.

Why does this area correspond to $n^2 + 12n + 36$?



Show me where n^2 is in this diagram. Where is $12n$? Where is the 36 part of the diagram?

Now show me why it also shows $(n + 6)^2$.
Where is the $n + 6$?

Ask students to identify groups of expressions that are equivalent and explain their reasoning. For example, E1 is equivalent to E10, E8 is equivalent to E9, and E4 is equivalent to E5.

Whole-class discussion (20 minutes)

Hold a whole-class interactive discussion to review what has been learned over this lesson.

Ask each group of students to justify, using their poster, why two expressions are equivalent.

Then use mini-whiteboards and questioning to begin to generalize the learning:

Draw me an area that shows this expression: $3(x+4)$

Write me a different expression that gives the same area.

Draw me an area that shows this expression: $(4y)^2$

Write me a different expression that gives the same area.

Draw me an area that shows this expression: $(z+5)^2$

Write me a different expression that gives the same area.

Draw me an area that shows this expression: $\frac{w+6}{2}$

Write me a different expression that gives the same area.

Follow-up lesson: improving individual solutions to the assessment task (10 minutes)

Return students' work on the assessment task *Interpreting Expressions,* along with a fresh copy of the task sheet. If you have not added questions to individual pieces of work, write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

Read through the solution you wrote [yesterday] and the questions (on the board/written on your script).

Answer the questions and then thinking about what you learned this lesson, write a new solution to see if you can improve your work.

Some teachers give this as a homework task.

SOLUTIONS

Assessment task: *Interpreting Expressions*

1a. $5n + 4$.

1b. $5(n + 4)$.

1c. $\frac{n+4}{5}$.

1d. $3n^2$.

1e. $(3n)^2$.

2. $2(n + 3) \neq 2n + 3$, $2(n + 3) = 2n + 6$.

$\frac{10n-5}{5} = 2n - 1$ is correct.

$(5n)^2 \neq 5n^2$, $(5n)^2 = 25n^2$.

$(n + 3)^2 \neq n^2 + 3^2$, $(n + 3)^2 = n^2 + 6n + 9$ *($n^2 + 3^2$ does however equal $n^2 + 9$)*.

Lesson task

This table is for convenience only: it is helpful **not** to refer to cards by these letters in class, but rather to the content of the cards.

Expressions	Words	Tables	Areas
E1 E10	W4 *W13 (Blank) Divide n by 2 then add 3*	T7	A5
E2	*W11 (Blank) Square n then multiply by 3*	T4	A3
E3 *E13 (Blank) 2(n + 6)*	W3 W7	T1	A1
E4 E5	W1 W5	T6	A2
E6	W8	T8	A6
E7 *E14 (Blank)* $9n^2$	W2 W10	T2	A4
E8 E9	W6 *W14 (Blank) Square n, add 12 multiplied by n, add 36*	T5	A7
E11	W9	T3	A8
E12	*W12 (Blank) Square n then add 6 squared*		

Interpreting Expressions

1. Write algebraic expressions for each of the following:

   a. Multiply n by 5 then add 4. ....

   b. Add 4 to n then multiply by 5. ....

   c. Add 4 to n then divide by 5. ....

   d. Multiply n by n then multiply by 3. ....

   e. Multiply n by 3 then square the result. ....

2. The equations below were created by students who were asked to write equivalent expressions on either side of the equals sign.

   Imagine you are a teacher. Your job is to decide whether their work is right or wrong. If you see an equation that is false, then:

   a. Cross out the expression on the right and replace it with an expression that is equivalent to the one on the left.

   b. Explain what is wrong, using words or diagrams.

$$2(n + 3) = 2n + 3$$

$$\frac{10n - 5}{5} = 2n - 1$$

$$(5n)^2 = 5n^2$$

$$(n + 3)^2 = n^2 + 3^2 = n^2 + 9$$

Card Set A: Expressions

E1 $$\frac{n+6}{2}$$	E2 $$3n^2$$
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E13	E14

Card Set B: Words

W1 Multiply n by two, then add six.	**W2** Multiply n by three, then square the answer.
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W5 Add three to n then multiply by two.	**W6** Add six to n then square the answer.
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W9 Square n, then add six	**W10** Square n, then multiply by nine
W11	**W12**
W13	**W14**

Card Set C: Tables

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T7

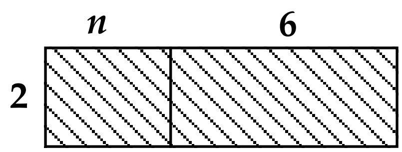
n	1	2	3	4
Ans		4		5

T8

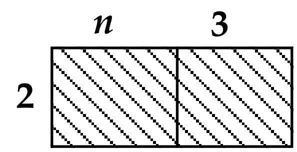
n	1	2	3	4
Ans	6.5	7	7.5	8

Card Set D: Areas

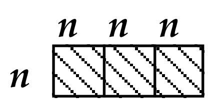
A1



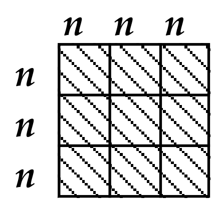
A2



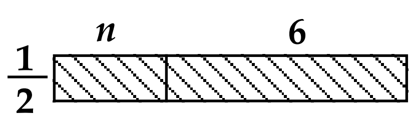
A3



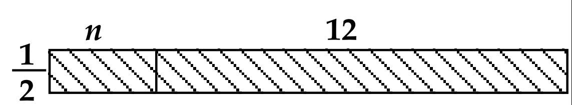
A4



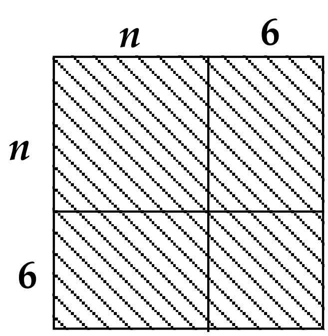
A5



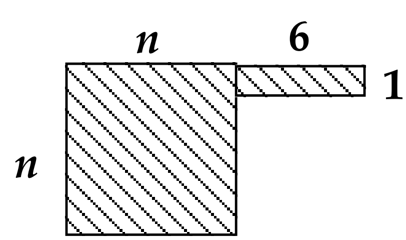
A6



A7



A8



Matching Expressions and Words

$4(n+2)$	Multiply n by two, then add four.
$2(n+4)$	Add four to n, then multiply by two.
$4n+2$	Add two to n, then multiply by four.

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org