CONCEPT DEVELOPMENT

Mathematics Assessment Project
CLASSROOM CHALLENGES
A Formative Assessment Lesson

Representing Inequalities Graphically

Representing Inequalities Graphically

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to use linear inequalities to create a set of solutions. In particular, the lesson will help you identify and assist students who have difficulties in:

- Representing a constraint by shading the correct side of the inequality line.
- Understanding how combining inequalities affects a solution space.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

A-REI: Represent and solve equations and inequalities graphically.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practice 1:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

This lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task that is designed to reveal their current levels of understanding and difficulties. You then review their work and create questions for students to answer in order to improve their solutions.

- After a whole-class introduction students work collaboratively, in pairs, on a game. One student decides on the position of a 'target point' on a coordinate grid and gives clues in the form of algebraic inequalities (e.g. $3y + 2x \le 12$). The other student uses these clues to find the location of the target point. There is a strategic element to the game: what are the best clues to give, so that the target may be found quickly?

- In a whole-class discussion students review the main math concepts of the lesson.

- In a follow-up lesson, students try to improve their original responses to the assessment task.

MATERIALS REQUIRED

- Each student will need two copies of the assessment task *Combining Inequalities*, the game sheet *Give Us a Clue!*, either a mini-whiteboard (preferably with a square grid) or a sheet of squared paper, a pen, and an eraser.

- There is a projector resource to support whole-class discussions.

TIME NEEDED

15 minutes before the lesson, one 70-minute lesson (or two shorter lessons), and 15 minutes in a follow-up lesson. Timings given are approximate.

BEFORE THE LESSON

Assessment task: *Combining Inequalities* (15 minutes)

Have students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the next lesson.

Give each student a copy of *Combining Inequalities*, a pencil, and a ruler.
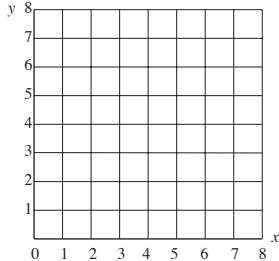
Briefly introduce the task:

> *Spend 15 minutes individually, answering these questions.*
>
> *Show all your work, so that I can understand your reasoning.*

It is important that students answer the questions without your assistance, as far as possible.

Students should not worry too much if they cannot understand or do everything, because in the next lesson they will engage in a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to answer questions such as these confidently. This is their goal.

Combining Inequalities

Some treasure has been buried at a point (x, y) on the grid, where x and y are whole numbers.

Here are three clues to help you find the treasure:

Clue 1: $x > 2$ **Clue 2:** $x + y < 8$ **Clue 3:** $2y - x \geq 0$

1. Which of the following points could be a possible location for the treasure?
   The points must satisfy *all three* clues.
   Circle the answers you choose.

   (3,2)   (2,3)   (5,3)   (3, 5)   (4,3)   (5, 2)

2. On the grid show *all* the possible places the treasure could be located.

3. Here are two more clues: **Clue 4:** $y > x - 4$ **Clue 5:** $y < x - 1$

   Which clue doesn't help at all? ....

   Explain why.

   ....

   ....

   At which point is the treasure located?

Assessing students' responses

Collect students' written work for formative assessment. Read through their papers and make informal notes on what their work reveals about their current levels of understanding.

We strongly suggest that you do not write scores on students' work. Research shows that this is counterproductive, as it encourages students to compare scores and distracts their attention from what they could do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit. We suggest you make a list of questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these questions on the board when you return the work to the students in the follow-up lesson.

Common issues:	Suggested questions and prompts:
Difficulty distinguishing between > and ≥ or < and ≤ For example: The student includes (2,3) and (5,3) as possible locations for the target. (Q1) Or: The student states the treasure is located at (2,1), (4,3), (5,3), or (3,2). (Q3) Or: The student does not use a dashed line for < or > inequalities. (Q2 or Q3)	• Write the inequalities into words. • What is the difference between > and ≥? • What is the difference between < and ≤? • The point (2,5) is outside the region where the treasure is located. Which clue tells you this? • Are points on the line $x = 2$ possible locations for the treasure? Are points on the line $2y - x = 0$ possible locations for the treasure? How can you distinguish graphically between the two? • Mark a point where the treasure is definitely not.
Guess and check used to figure out the possible location for the treasure (Q2) For example: The student does not draw the inequality boundaries as lines on the grid, but instead guesses possible locations for the treasure and checks to see if they fit the clues.	• Can you think of a quicker way to figure out the possible locations? • How can you convince me there are no other possible points? • How can you use the graph to show the region where the treasure is located?
Insufficient reasoning provided (Q3) For example: The student does not explain the reason why Clue 4 is unhelpful.	• How does the clue affect the region where the treasure is located? • Does this clue help you find the position of the treasure?
Assumption made in Q3 that the treasure is located at one of the points chosen in Q1	• Check to see if your point fits your new clue.
All answers are correct The student needs an extension task.	• Another treasure is at (6,5). Write just two clues that will locate the exact position of the treasure. Your clues should use the inequality symbols >, <, ≥, or ≤.

SUGGESTED LESSON OUTLINE

If you have a short lesson, or you find the lesson is progressing at a slower pace than anticipated, we suggest you end the lesson after the paired work, 'Preparing to play *Give Us a Clue!'* and continue in a second lesson.

Whole-class interactive introduction: *Hunting the Target* (15 minutes)

Give each student either a mini-whiteboard, pen, and eraser, or a sheet of squared paper.

Use Slide P-1 of the projector resource*s* to project the 6 × 6 coordinate grid on to the board.

Write the pair of coordinates (2,2) on a piece of paper, fold it in half (hiding the coordinates) and stick this to the board.

> *I am thinking of a target point on this grid. I have written the coordinates on this paper. Both coordinates are integers.*
>
> *Your task is to guess which point I am thinking of.*
>
> *Here is the first clue: 3y + 2x ≤ 12*
>
> *Does anyone know what this clue means?*

Students may need careful leading through this idea, so take this stage slowly. Use questions such as the following, asking students to respond using their mini-whiteboards:

> *Show me the coordinates of a point that satisfies the clue.*
>
> *Can you show me another point? … and another? How do you know?*

As students suggest possible points, mark these clearly on the grid.

> *Where are all the points that satisfy this clue?*
> *[On or below the line 3y + 2x = 12.]*
>
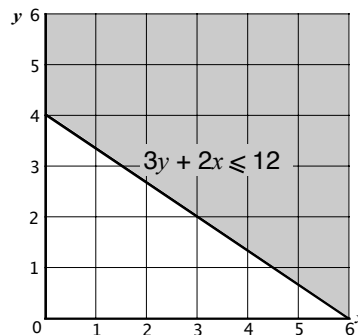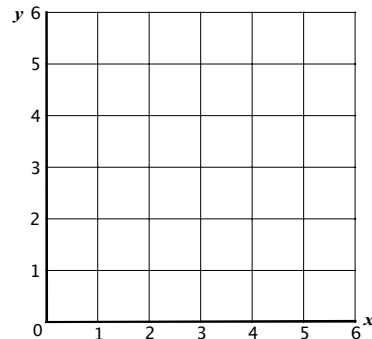> *Where are all the points that don't satisfy this clue?*
> *[Above the line 3y + 2x = 12.]*
>
> *Give me a point that **just** satisfies the clue.*
> *Give me a point that **easily** satisfies the clue.*

Explain that for this lesson, the region that does not satisfy a clue is to be shaded out.

To help students keep track of each clue you may want to use a different color marker for each inequality.
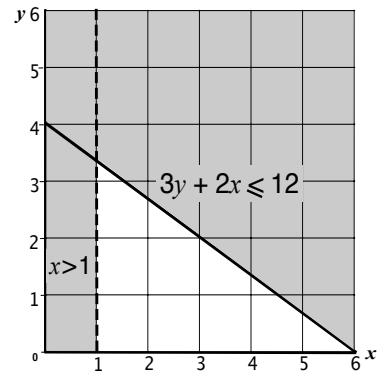
Here's the second clue: x > 1

Shade out all the points that are eliminated.

Show me the new region.

Which points are possible now?

Is (1,2) a possible point?

Explain that we use a dashed line to show that the points on the line $x = 1$ are not included as possible points for the target.

Here's the third clue: y > x − 1

Shade out all the points that are eliminated.
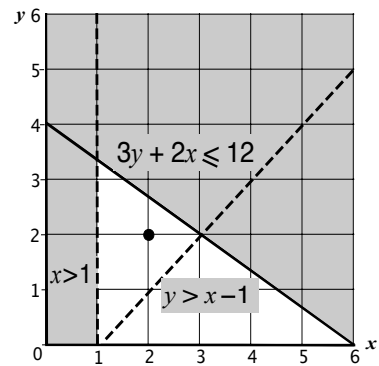
Which points are possible now?

Show me the new region.

Which points are possible now? How do you know?

Is (2,2) the only possibility?

Why can't (3,2) be a possible point for the target?

Although there are many non-integer points that are possible, explain that for this lesson we will stick with integer coordinates.

Preparing to play *Give Us a Clue!* (10 minutes)

Give each student a copy of the sheet *Give Us a Clue!*

Use Slide P-2 of the projector resource to project the 8 × 8 coordinate grid onto the board. You are soon going to play a game called 'Give Us a Clue!' Which points are possible now?

You will use the lines on the small graphs on the handout.

Before beginning the game you need to figure out the inequalities for the regions to the left and right of each given line. You will use these inequalities as clues in the game.

For example, look at the line 2x − y = 8.

Which side of the line are points that fit the inequality $2x − y \geq 8$?

Which side of the line are points that fit the inequality $2x − y \leq 8$?

In order to answer these two questions, it is helpful to test the inequality with specific pairs of coordinates. These are sometimes called test points.

(0,0) is usually a good choice for a test point, since it makes the arithmetic easy, but if the line itself goes through the origin, then another point should be chosen:
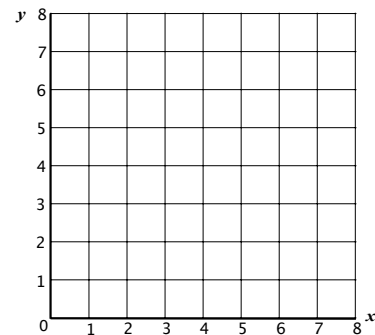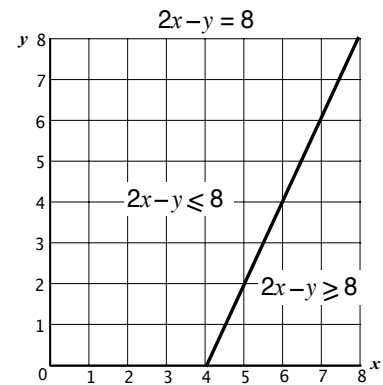
Can you put the inequality into words?

Let's use the origin (0,0) as a test point. This point is to the left of the line.

Which of the two inequalities [2x − y ≥ 8 or 2x − y ≤ 8] does it fit?

Now choose your own test point to the right of the line.
Use its coordinates to check the inequality for this region.

Since $2(0) - 0 \le 8$ is true, the origin is included in the region $2x - y \le 8$. This region is to the left of the line.

Paired work: preparing to play *Give Us a Clue!* (10 minutes)

Organize the class into pairs of students.

Use Slide P-3 of the projector resource to explain how students should work collaboratively:

> ### Preparing to play the game
>
> 1. Take turns to figure out the inequalities for each region of the twelve small graphs.
>
> 2. Once you have done this, explain to your partner how you came to your decision.
>
> 3. Your partner should either explain that reasoning again in his or her own words, or challenge the reasons you gave.
>
> 4. You need to agree on and **both** be able to explain the inequalities for each region of each graph.
>
> 5. Make sure you write all the inequalities on your own copy of *Give Us a Clue!*
>
> There is no need to shade the graphs.

The purpose of this structured paired work is to make each student engage with their partner's explanations and to take responsibility for their partner's understanding.

You have two tasks during the paired work: to note aspects of the task students find difficult and support student reasoning.

Note aspects of the task students find difficult

For example, are students having difficulties using a test point? Do they understand the difference between inequality symbols? You can use information about particular difficulties to focus the whole-class discussion at the end of the lesson.

Support student reasoning

Try not to make suggestions that move students towards a particular answer. Instead, ask questions to help students to reason together. For students struggling to understand the symbols, it may help if they put the inequalities into words.

How did you figure out the inequality for this region?

[Select a graph that goes through the origin.] Why is (0,0) not a good test point for this graph?

[Select one of the first four graphs.] Why is (4,4) not a good test point to use for this graph?

Sharing work: preparing to play *Give Us a Clue!* (10 minutes)

Ask students to check their work with a neighboring pair of students.

Check to see which graphs are different.

When there is a disagreement, take turns to justify your decision. If you still don't agree, ask for further explanation.

Both of you need to agree and understand the math.

Students playing *Give Us a Clue!* (15 minutes)

When students are satisfied with their twelve graphs, use Slide P-4 of the projector resource to introduce the game:

> ## Playing the Game
>
> 1. One of you will be the target picker and the other the target hunter.
>
> 2. The target picker decides on the position of the target and gives the clues.
>
> 3. When giving clues, the target picker can use any inequality sign ($\leq$, $<$, $\geq$, $>$), but not the '=' sign. Try to give helpful clues!
>   As you give the clues, write them as a list on your mini-whiteboard.
>
> 4. The target hunter uses the clues to find the target.
>
> 5. The aim of the game for both partners, is to find the target in the least number of tries.
>
> 6. Both partners must use a blank grid to keep track of the clues that are given
>
> 7. Each time a clue is given, shade out the region where the target cannot be located.

It is important that students cannot see each other's graphs. They could use a book or folder to hide the graph from their partner.

Encourage students to give clues using the correct inequality language, rather than using imprecise language such as "The point is above the line."

When the target picker has used all the useful inequalities on the handout, they could make up their own.

At the end of each game, students should check each other's graphs. If they are not the same, encourage them to work together to identify mistakes made. The mini-whiteboard listing the clues may help sort out disagreements. This should be seen as a collaborative rather than competitive activity.

Then students can reverse roles.

For students who have successfully completed this task, ask them to create their own inequalities and use them to play the game with their partner.

Whole-class discussion (10 minutes)

In the summary discussion you can explore the best strategy for giving a clue, while revising the main math concepts in the lesson. Students should use their mini-whiteboards to respond to your questions.

Use Slide P-2 of the projector resource to project the 8×8 coordinate grid on to the board.

We will now investigate how to give the best clues for targets within an 8 × 8 grid. [By best I mean clues that will give the target coordinates quickly.] We are still using the inequalities on the work sheet.

Can anyone think of the best first clue for the point (2,5)? [y > 2x.]

Ask a few students to justify their answers. Use different color markers to draw their clues on the board.

In this case, is the clue y ≥ 2x better than the clue y > 2x? Why?

Once students are satisfied that they are using the best first clue ask:

What is the next best clue? [y < x + 4.]

Again ask students to explain their reasoning.

How do you know y < x + 4 is a better clue than y < x? Show me.

If students are struggling with the difference between a clue that uses < and one that uses ≤ ask:

How many points could the target be if you use the clue y < x + 4?

How many points could the target be if you use the clue y ≤ x + 4?

Now ask for a final clue:

And what is another good clue? [y > 4]

How many places could the target be now?
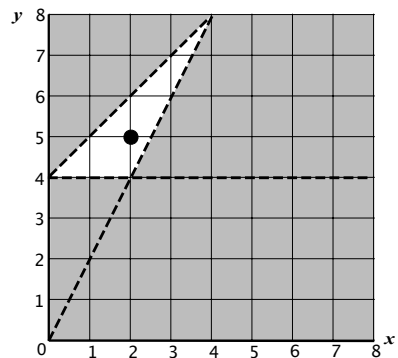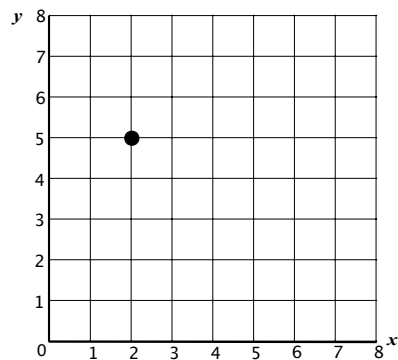
You could extend this further by asking:

Can you think of a target point within the 8 × 8 grid that only requires two clues?

It will be a point on a line. For example, when the target point is (8,4), the clues could be $y \leq 4$ and $x + 2y \geq 16$.

Follow-up lesson: reviewing the assessment task (15 minutes)

Return their original assessment *Combining Inequalities* to the students, as well as a second blank copy of the task. If you have not added questions to individual pieces of work or printed a list of questions, then write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

Look at your original responses and the questions (on the board/written on your scripts).

Answer these questions and revise your response.

Explain to students that Questions 1 and 2 are concerned with just the first three clues. When answering these questions they should ignore Clues 4 and 5.
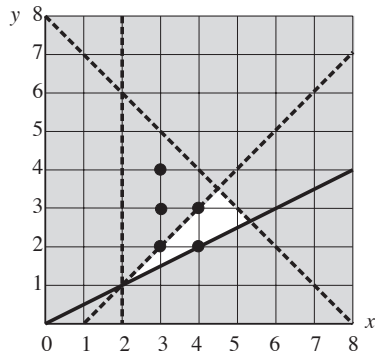
SOLUTIONS

Assessment task: *Combining Inequalities*

1. The points (3,2) and (4,3) are the points that satisfy all clues.

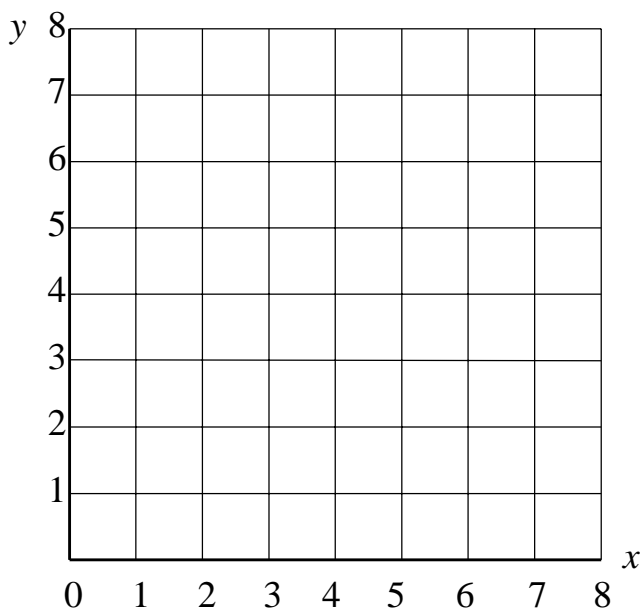2. The possible places for the target are indicated by the bold dots on the grid below:
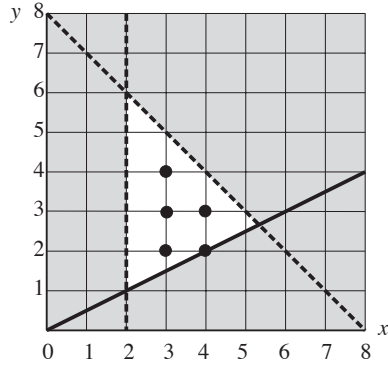
3. Clue 4 is unhelpful because it doesn't add any extra information.

   This is because the region $y > x - 4$ includes all of the above region.

   Clue 5 excludes all solutions but (4,2) so this is where the treasure is located. (See diagram)

Combining Inequalities

Some treasure has been buried at a point (***x,y***) on the grid, where ***x*** and ***y*** are whole numbers.

Here are three clues to help you find the treasure:

Clue 1: $x > 2$ **Clue 2:** $x + y < 8$ **Clue 3:** $2y - x \geq 0$

1. Which of the following points could be a possible location for the treasure?

   The points must satisfy *all three* clues.

   Circle the answers you choose.

   (3,2)   (2,3)   (5,3)   (3, 5)   (4,3)   (5, 2)

2. On the grid show *all* the possible places the treasure could be located.

3. Here are two more clues: **Clue 4:** $y > x - 4$ **Clue 5:** $y < x - 1$

   Which clue doesn't help at all? --

   Explain why.

   At which point is the treasure located?

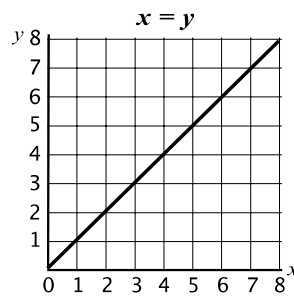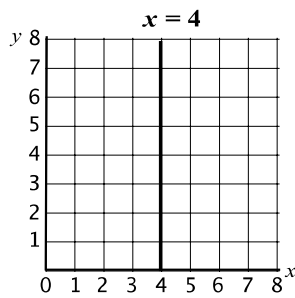
Give Us a Clue!

Use these grids to record the clues given by the teacher or your partner.

Game 1

y 8
7
6
5
4
3
2
1
0 1 2 3 4 5 6 7 8 x

Game 2

y 8
7
6
5
4
3
2
1
0 1 2 3 4 5 6 7 8 x
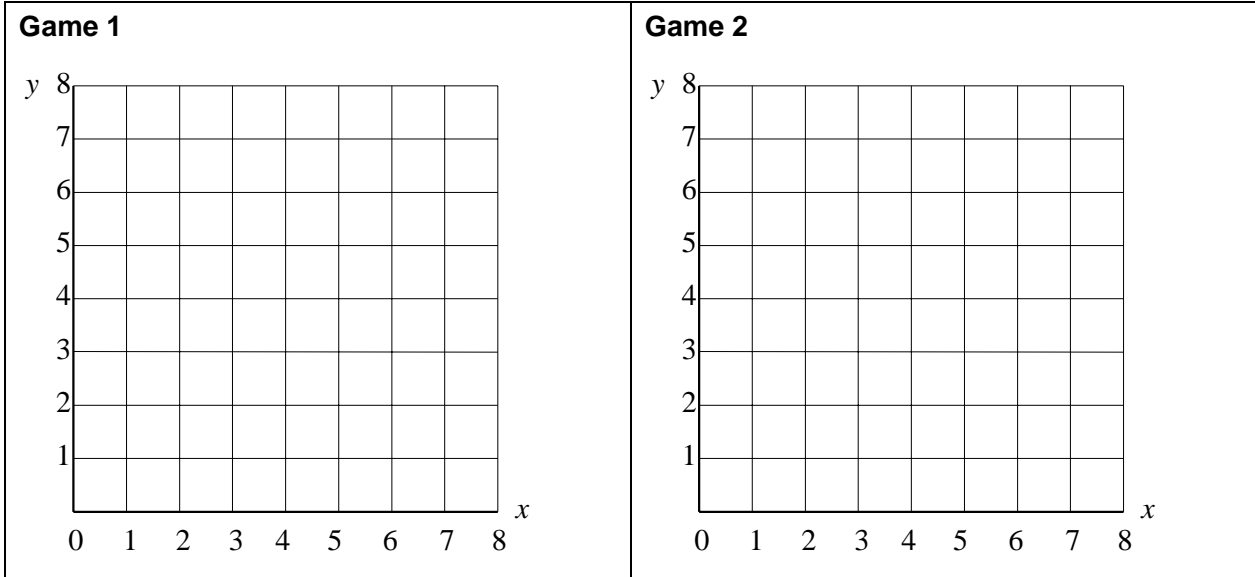
Use these graphs to invent your questions.

$y + x = 8$

$x = 4$

$x = y$

$y = 4$

$y = x + 4$

$y + 4 = x$

$y + x = 4$

$x + y = 12$

$y = 2x$

$2x - y = 8$

$x + 2y = 16$

$2y = 8 - x$

Representing Inequalities Graphically
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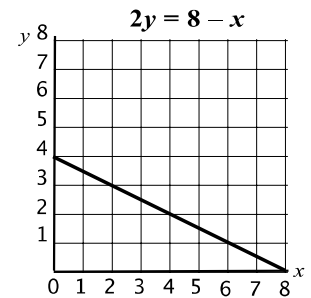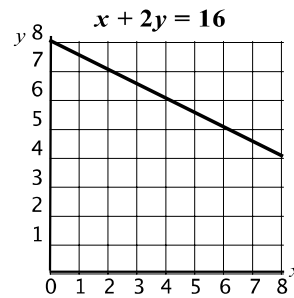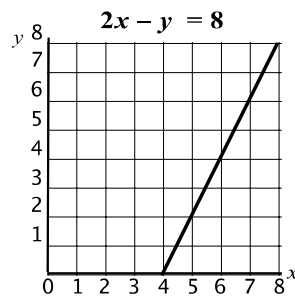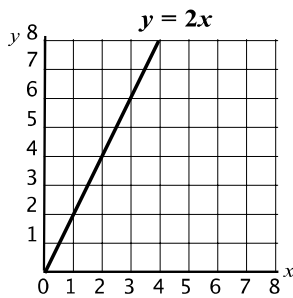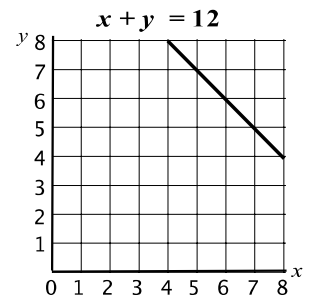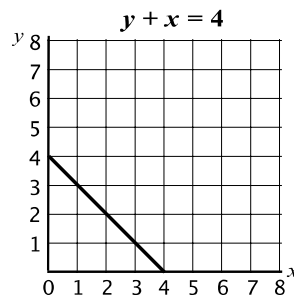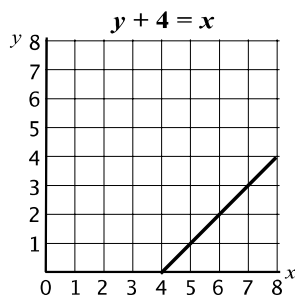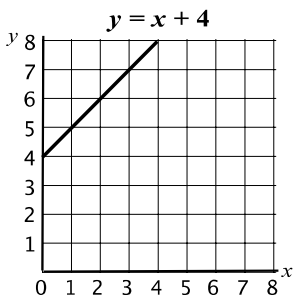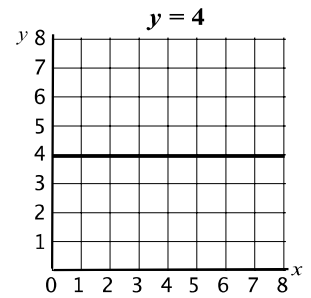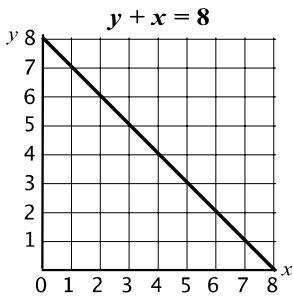
Hunting the Target
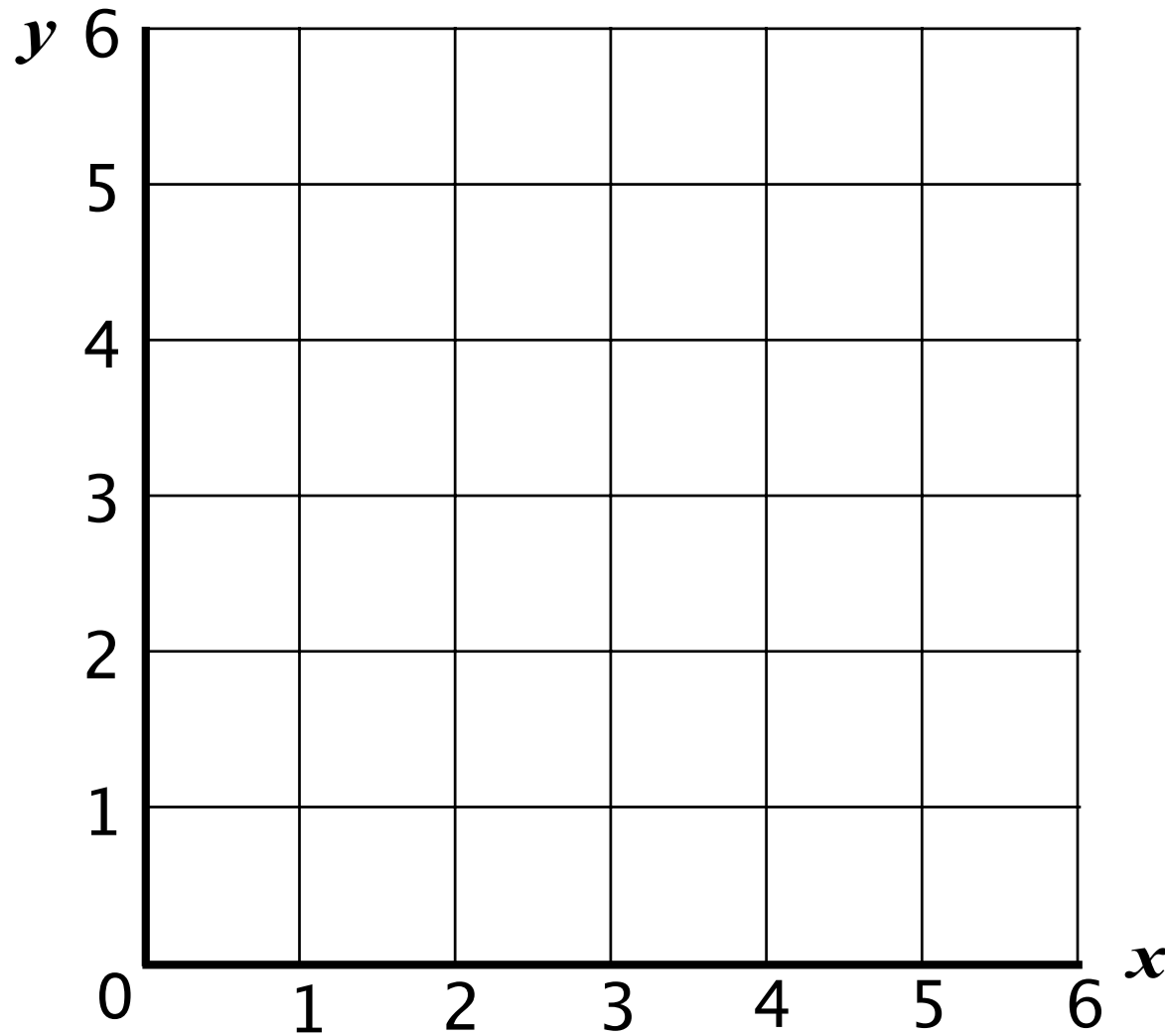
Give Us a Clue
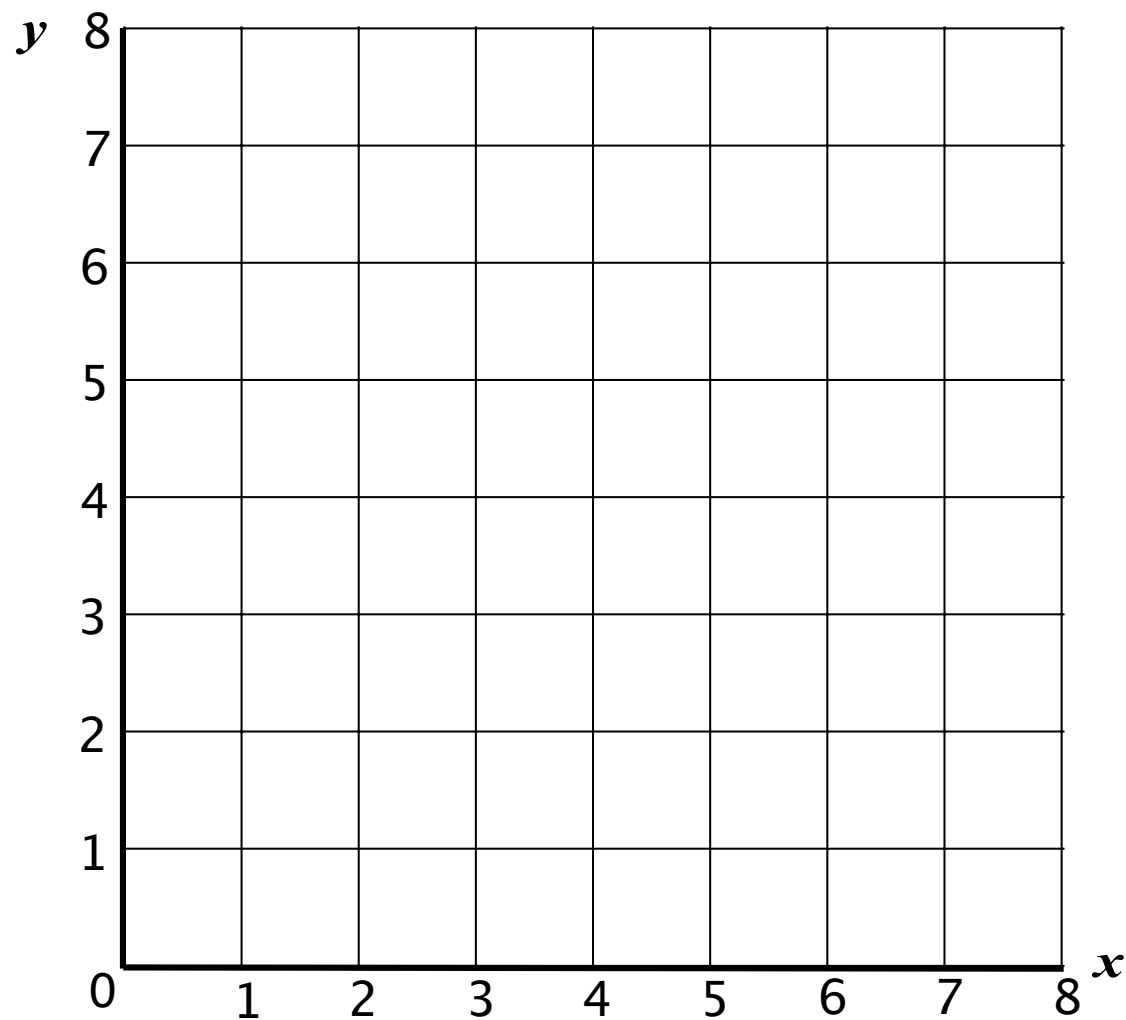
Preparing to play the game

1. Take turns to figure out the inequalities for each region of the twelve small graphs.

2. Once you have done this, explain to your partner how you came to your decision.

3. Your partner should either explain that reasoning again in his or her own words, or challenge the reasons you gave.

4. You need to agree on and **both** be able to explain the inequalities for each region of each graph.

5. Make sure you write all the inequalities on your own copy of *Give Us a Clue!*

There is no need to shade the graphs.

Playing the Game

1. One of you will be the target picker and the other the target hunter.

2. The target picker decides on the position of the target and gives the clues.

3. When giving clues, the target picker can use any inequality sign ($\leq$, $<$, $\geq$, $>$), but not the '=' sign. Try to give helpful clues!
   As you give the clues, write them as a list on your mini-whiteboard.

4. The target hunter uses the clues to find the target.

5. The aim of the game for both partners, is to find the target in the least number of tries.

6. Both partners must use a blank grid to keep track of the clues that are given.

7. Each time a clue is given, shade out the region where the target cannot be located.

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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with
Hugh Burkhardt, Rita Crust, Andy Noyes, and **Daniel Pead**

We are grateful to the many teachers and students, in the UK and the US,
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Alan Schoenfeld at the University of California, Berkeley, and
Hugh Burkhardt, Daniel Pead, and **Malcolm Swan** at the University of Nottingham

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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org