CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Evaluating Statements About Length and Area

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Evaluating Statements About Length and Area

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students can:

- Understand the concepts of length and area.
- Use the concept of area in proving why two areas are or are not equal.
- Construct their own examples and counterexamples to help justify or refute conjectures.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

G-CO: Prove geometric theorems.
G-SRT: Prove theorems involving similarity.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 1, 3, 6, and 7:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

Students will need to be familiar with the concepts of area and perimeter. This activity will help consolidate understanding and overcome some common misconceptions such as the notion that perimeter and area are in some way related.

The unit is structured in the following way:

- Before the lesson, students work individually on an assessment task designed to reveal their current understanding and difficulties. You then review their work and create questions for students to answer in order to improve their solutions.
- In a whole-class introduction students critique examples of other students' work, modeling the level of reasoning that is required in the main part of the lesson where students are grouped and given some mathematical statements. Using their own examples, counterexamples, and arguments students decide if the statements are true or false, creating posters showing their collaborative reasoning.
- In a whole-class discussion, students explain and compare the solution strategies they have used.
- In a follow-up lesson, students return to the original assessment task and improve their responses.

MATERIALS REQUIRED

- Each student will need two copies of *Shape Statements*, some plain paper, and a copy of *Student Work: Diagonals of a Quadrilateral*.
- Each pair of students will need *Card Set A: Always, Sometimes, or Never True*? (cut up into cards), a large sheet of paper, and a glue stick.
- You will also need several copies of *Card Set B: Some Hints* (cut up into cards.)

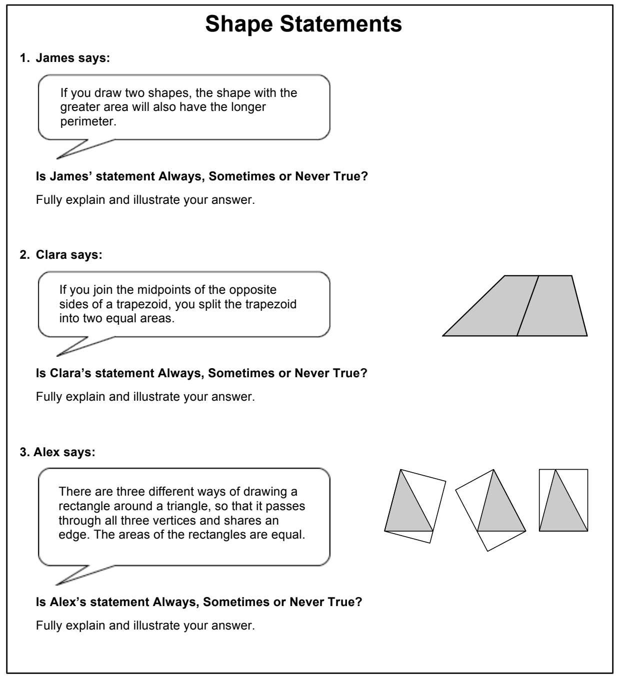
TIME NEEDED

15 minutes before the lesson, a 100-minute lesson (or two 50-minute lessons) and 10 minutes in a follow-up lesson. Timings are approximate. Exact timings will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Shape Statements* (15 minutes)

Have students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work, to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.



Shape Statements

1. James says:

If you draw two shapes, the shape with the greater area will also have the longer perimeter.

Is James' statement Always, Sometimes or Never True?

Fully explain and illustrate your answer.

2. Clara says:

If you join the midpoints of the opposite sides of a trapezoid, you split the trapezoid into two equal areas.

Is Clara's statement Always, Sometimes or Never True?

Fully explain and illustrate your answer.

3. Alex says:

There are three different ways of drawing a rectangle around a triangle, so that it passes through all three vertices and shares an edge. The areas of the rectangles are equal.

Is Alex's statement Always, Sometimes or Never True?

Fully explain and illustrate your answer.

Give each student a copy of *Shape Statements* and some plain paper to work on.

> *Spend 15 minutes working individually, answering these questions on the plain paper provided.*

It is important that, as far as possible, students are allowed to answer the questions without your assistance.

Students should not worry too much if they cannot understand or do everything because in the next lesson they will engage in a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to be able to answer questions such as these confidently. This is their goal.

Assessing students' responses

Collect students' written work and read through their responses. Make some informal notes on what their work reveals about their current levels of understanding and their different problem-solving approaches.

We suggest that you do not write scores on students' work. The research shows that this is counterproductive as it encourages students to compare scores, and distracts their attention from what they are to do to improve their mathematics.

Instead, help students to make further progress by asking questions that focus attention on aspects of their work. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students in the follow-up lesson.

Common issues:	**Suggested questions and prompts:**
Makes incorrect assumptions For example: The student assumes that the shape with the greater area has the greater perimeter (Q1). Or: The student assumes that only congruent shapes have equal areas (Q2).	• Do shapes with greater perimeters have to have greater areas? How could you check? • Do shapes need to be congruent to have equal areas?
Fails to consider specific examples For example: The student does not create shapes with different dimensions to support or disprove a statement.	• Think of a shape. Think of some dimensions for this shape. Is the statement true for this shape? • Can you think of another shape for which the statement is not true?
Fails to consider a wide enough range of examples For example: The student assumes that a specific example is sufficient to prove a conjecture. Or: The student does not consider extreme cases (Q1, Q2, and Q3).	• Can you draw some other examples to test the statement? • How do you know your answer is correct for all shapes? • Does the statement work for obtuse angles? (Q3)
Fails to draw construction lines (Q2 and Q3)	• Can you split the shape into smaller shapes to help test the statement? • Can you add lines to the shape to help test the statement?
Fails to apply algebraic reasoning For example: The student cannot see the relevance of the formula for the area of a trapezoid or triangle (Q2 and Q3).	• Can you use an area formula to support your answer?

SUGGESTED LESSON OUTLINE

Interactive whole-class introduction (20 minutes)

This activity is best introduced with a whole-class discussion, in which you model the reasoning required for the lesson's activities.

Give each student a piece of plain paper. Use Slide P-1 of the projector resource or write this statement on the board:

> **Diagonals of a Quadrilateral**
> If you draw in the two diagonals of a quadrilateral, you divide the quadrilateral into four equal areas.

Is this statement always, sometimes or never true?

If you think the statement is always true or never true, then how would you convince someone else?
If you think the statement is sometimes true, would you be able to identify all the cases of a quadrilateral where it is true/not true?

Allow students a few minutes to think about this as individuals. It is important that they understand the problem and spend some time thinking about what would need to be included in their solutions. After a few minutes working individually, give students a couple of minutes to discuss their initial ideas in pairs.

Bring the class together and discuss the statement.

What kinds of quadrilateral could we look at?

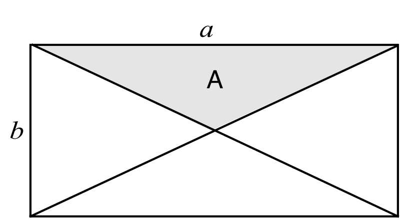
Students may not have considered all possible quadrilaterals, or may have misconceptions about some of the properties of quadrilaterals, so this could form part of your discussions.

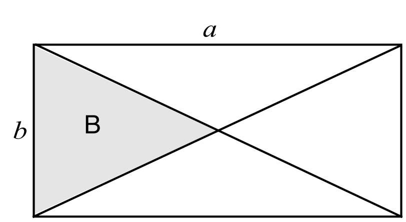
Can anyone suggest a type of quadrilateral for which the result is true?
Can anyone suggest a type of quadrilateral for which the result is false?

Students may suggest squares, rectangles or parallelograms as quadrilaterals for which the result is true and kites, trapezoids and so on as quadrilaterals for which the result is false. They are likely to have come to their conclusions by calculating the areas for a few different sized quadrilaterals. It is important that students move beyond this and look at the general case. They should be encouraged to justify their conjectures.

OK, it seems like this result works for some quadrilaterals but not for others. So it is sometimes true. How do we show that it will always work/not work for a specific quadrilateral? For example, in a rectangle, how could you convince me that the four triangles will always have the same area?

Students may use a method similar to this one:





To show that the two pairs of triangles have the same area, we use the area formula
$\frac{1}{2} \times \text{base} \times \text{height}$

Area of triangle A $= \frac{1}{2} \times a \times \frac{1}{2}b = \frac{1}{4}ab$ Area of triangle B $= \frac{1}{2} \times b \times \frac{1}{2}a = \frac{1}{4}ab$

We can therefore conclude that this statement is true for all rectangles.

You may now want to ask:

This method has assumed that the two diagonals bisect each other. How do we know this? Show me. [Students could then prove two opposite triangles are congruent (using Angle, Side, Angle).]

Throughout this lesson, it is important that students recognize that there are different methods that can be used to solve a problem. During the whole-class discussion they will consider which methods they prefer, as well as spending some time in the next activity evaluating different approaches to a problem.

Individual task: *Student Work: Diagonals of a Quadrilateral* (20 minutes)

Give each student a copy of *Student Work: Diagonals of a Quadrilateral* containing sample work from two students. The sample solutions allow students to evaluate two different strategies used to justify or disprove the statement.

I want you to assess this work. Write comments on the sheet.

The sample work focuses on justifying/disproving the conjecture for two types of quadrilateral: kites and parallelograms. The aim of this activity is to encourage students to consider what makes a good explanation, as well as addressing some common misconceptions concerning length and area.

To encourage students to write more than simple comments such as ‘great work’ or ‘incorrect’, display Slide P-2 of the projector resource containing some questions for them to think about as they evaluate the two pieces of sample work:

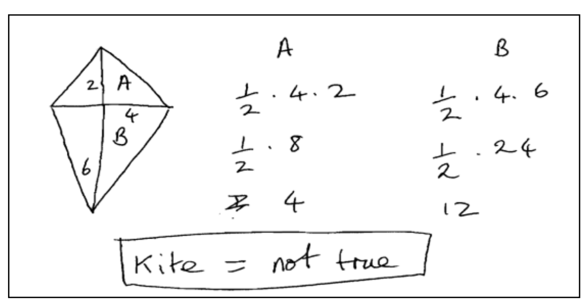
Student Work: Diagonals of a Quadrilateral

- What do you like about the work?
- Has this student made any incorrect assumptions?
- Is the work accurate?
- How can the explanation be improved?

Student Work 1

In this example, the areas of the two triangles A and B have been calculated correctly. The work would benefit from more explanation.

The student has concluded the statement is never true after only considering one specific example. This is incorrect; for example, the statement is true for a rhombus.



This strategy can help students get started on the task and can be useful to show that a given statement is sometimes true.

When is this method useful? [It shows the statement is sometimes true.]

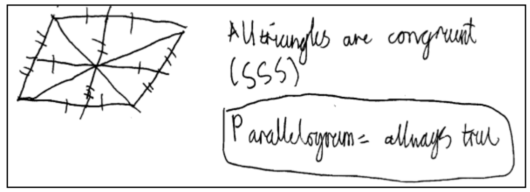
Show me a kite that makes the statement false.

Students often, mistakenly, think this method can be used to show a statement is always true or always false.

Student Work 2

In this example, the student has indicated lines of equal length. Some justification for this is needed.

The student has made the incorrect assumption that the diagonals are always equal. The student has then incorrectly concluded that all the small triangles are congruent.

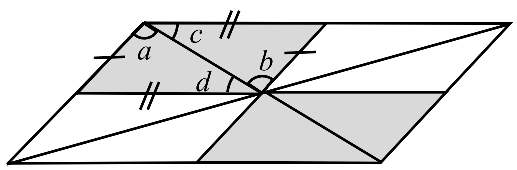


What mistake has this student made?

Is the statement always, sometimes, or never true? Show me.

In this case the student's explanation is inaccurate.

In the parallelogram below all the white triangles are congruent and all the grey triangles are congruent.



This congruency can be proved using Side, Side, Side (SSS) or Angle, Side, Angle (ASA):

The diagonals of the larger parallelogram bisect each other. This means the small grey quadrilateral is a parallelogram. Opposite sides of the grey parallelogram are equal.

Angle a = Angle b and Angle c = Angle d (the diagonal of the parallelogram cuts two parallel lines.) The two grey triangles have one common side.

Therefore the four triangles consisting of one grey and one white triangle are all equal in area. Hence the statement is true.

This example shows how students can add construction lines in order to prove or disprove a statement.

Whole-class discussion: *Student Work: Diagonals of a Quadrilateral* (10 minutes)

Once students have assessed the sample work, discuss the different approaches used.

You may find it helpful to display the projector resources Slide P-3 *Student Work 1* and Slide P-4 *Student Work 2* to support the discussion.

Discussing each piece of sample work in turn, ask students to respond to the questions posed:

What do you like about the work?

Is the work accurate?

Has this student made any incorrect assumptions?

How can the explanation be improved?

Encourage students to evaluate the two pieces of sample work in terms of both their accuracy and quality of explanation. Ask students which of the two approaches they prefer and why.

Extending the lesson over two days

If you are taking two days to complete the unit you might want to end the first lesson here. Then, at the start of the second day, briefly remind students of their previous work before asking them to work on the collaborative task.

Collaborative small-group work (30 minutes)

Ask students to work in pairs or threes. Give each small group cut-up *Card Set A: Always, Sometimes, or Never True?,* a large sheet of paper for making a poster, and a glue stick.

You are going to choose ***one*** *statement to work on. Decide if it is always, sometimes, or never true.*

First, you will need to make sure that you fully understand the problem. It may help to try a few examples to begin with, before justifying or disproving the statement.

If the statement is sometimes true, you will need to explain in which cases it is true/false.

Students who you think may struggle should be guided towards statement Card A.

If students are successful in classifying their chosen statement, they should be encouraged to consider alternative methods, before moving on to a second statement card as appropriate.

Card F may provide more of a challenge for students who are progressing well.

You have two tasks during the small-group work: to note different student approaches to the task and to support student reasoning.

Note different student approaches to the task

Listen and watch students carefully. Note different student approaches to the task and any common mistakes. You can then use this information to focus a whole-class discussion towards the end of the lesson.

Support student reasoning

Try to avoid explaining things to students. Instead, encourage them to explain to one another and to you. If you find one student has produced a solution for a particular statement, challenge another student in the group to provide an explanation.

John thinks this statement is sometimes true. Sharon, why do you believe John thinks this?

If you find students have difficulty articulating their solutions, the cards from *Card Set B: Some Hints* can be used to support your own questioning of students, or can be handed out to students who are struggling.

If the whole-class is struggling on the same issue, you may want to write a couple of questions on the board and hold an interim, whole-class discussion.

Whole-class discussion (20 minutes)

There will not be time to go through every statement card (and it is likely that some cards will not have been attempted by any of the small groups of students), but a discussion about the chosen category for at least one of the statement cards is essential, focusing on reasoning and the different methods of justification employed by the students. It may be helpful to do a quick survey of the statement cards chosen by students and select one that has been attempted by more than one group.

For the chosen statement, ask each group that has attempted it to describe to the rest of the class their method for categorizing the statement as always, sometimes or never true. Then ask other students their views on which reasoning method is easiest to follow, as well as contributing ideas of alternative approaches. It is important that students consider a variety of methods and begin to develop a repertoire of approaches.

How else could we explain our reasoning for categorizing this statement as always/sometimes/never true?

Which explanation did you prefer? Why?

Is their explanation sufficient? How could we improve it?

Follow-up lesson: improving individual solutions to the assessment task (10 minutes)

Return the original assessment task *Shape Statements* to the students, together with a second, blank copy of the task.

If you have not added questions to individual pieces of work, write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

Look at your original response and the questions (on the board/written on your script).

Answer these questions and using what you have learned, revise your response.

Some teachers give this for homework.

SOLUTIONS

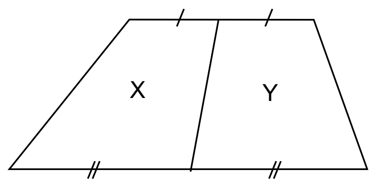
The solutions below show just one or two strategies for each statement. Students may think of other correct approaches.

Assessment task: *Shape Statements*

1. The first statement is **sometimes true**. The student should give examples of where it is true and also where it is not. For example, it is true if the shapes are two different sized circles, but it is not true if, one shape is a 1 by 100 rectangle and one is a 50 by 50 square.

2. The second statement is **sometimes true**. It is true if the sides joined are the parallel ones.

   This is shown in the diagram:

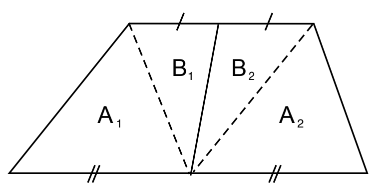


   Students may use the area of a trapezoid formula:

$$\frac{1}{2} \times \text{(sum of parallel sides)} \times \text{perpendicular distance between them}$$

   The sum of the parallel sides for trapezoid X and trapezoid Y are equal. Therefore, Area X = Area Y

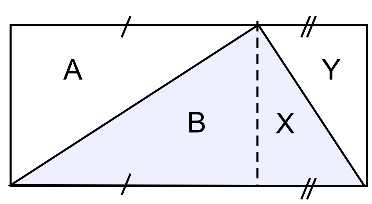
   Alternatively, students may split the trapezoid as shown:



   The two areas marked A_1 and A_2 are equal as they are triangles with the same height and base length. The two areas marked B_1 and B_2 are also equal as they are triangles with the same height and base length.

   Therefore, Area A_1 + Area B_1 = Area A_2 + Area B_2, so the trapezoid has been split into two equal areas.

3. The third statement is **sometimes true**. It is true for acute-angled triangles: the triangle's area is one half of the rectangle area in each case. This is shown in the diagram below:



   Area A = Area B and Area X = Area Y (diagonals of a rectangle cut the rectangle in half). The area of the rectangle is always twice as big as the area of the triangle.

   However, if a triangle has an obtuse angle it is not possible to draw three rectangles in the way described above.

Group work: Always, Sometimes, or Never True?

A. Cutting shapes: (a) Always true (b) Sometimes true.

This task is intended to help students to overcome the misconception that areas and perimeters are related in some way.

(a) Clearly, it is always true that cutting a piece off a shape will result in a decrease in area.

(b) On the hint cards, examples are shown where cutting a piece off can reduce, leave unchanged, or increase the perimeter.

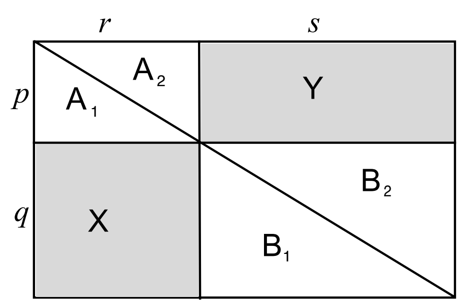
B. Sliding a triangle: (a) Always true (b) Always true.

(a) The base is always the same. The height will always be the same, because it is sliding along a line parallel to the base. That means the triangle will always have the same area as the area of a triangle is 'half the base times the height'.

(b) By sliding the top vertex to the right, we can make the perimeter as large as we like, while preserving the area.

C. Rectangles: (a) Always true (b) Sometimes true.

(a) This is most easily seen by labeling equal areas:



Area $A_1 = \frac{1}{2} \times r \times p$ and Area $A_2 = \frac{1}{2} \times p \times r$. Therefore, Area A_1 = Area A_2

Area $B_1 = \frac{1}{2} \times s \times q$ and Area $B_2 = \frac{1}{2} \times q \times s$. Therefore, Area B_1 = Area B_2

The diagonal cuts the rectangle in half:

Area A_1 + Area X + Area B_1 = Area A_2 + Area Y + Area B_2.

Therefore Area X = Area Y.

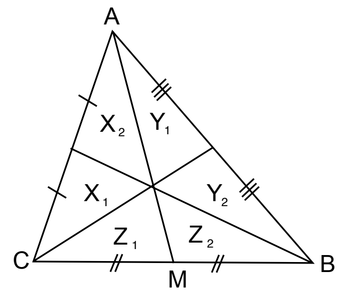
(b) The perimeters are equal when the two shaded rectangles have a common vertex at the center of the large rectangle:

Perimeter X = $2q + 2r$ and Perimeter Y = $2s + 2p$

The two perimeters are equal when $q + r = s + p$. This is true when $q = p$ and $s = r$.

D. Medians of triangles: Always true.

Since each of the medians bisects one side of the triangle, we can see that pairs of small triangles (labeled X_1, X_2; Y_1, Y_2; and Z_1, Z_2) have the same base length and height and so have equal areas.



Triangles AMB and AMC also have equal areas, therefore:

Area X_2 + Area X_1 + Area Z_1 = Area Y_1 + Area Y_2 + Area Z_2.

But we know that Area X_1 = Area X_2 and Area Y_1 = Area Y_2 and Area Z_1 = Area Z_2, so:

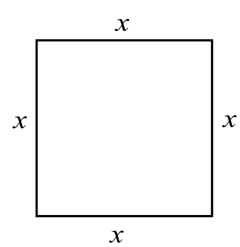
$2 \times$ Area X_1 = $2 \times$ Area Y_1

Area X_1 = Area Y_1

Therefore, all 6 triangles having the same area.

E. Square and circle: Never true.

Suppose the square has a perimeter of $4x$ units, which implies the area is x^2 units^2.



If the circle also has a perimeter of $4x$ units,

$2\pi r = 4x$

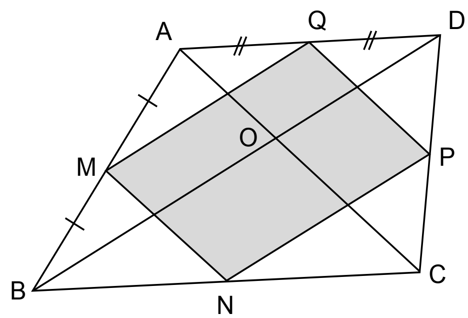
$\Rightarrow r = 2x/\pi$

$\Rightarrow \pi r^2 = 4x^2/\pi > 1$

So the circle has a greater area.

F. Midpoints of a quadrilateral: Always true.

Let ABCD be the quadrilateral and M, N, P, Q be the midpoints of the sides. Draw the diagonals AC and BD.



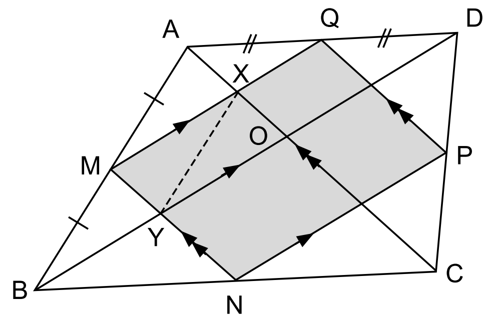
AMQ and ABD are similar triangles (angle BAD is a common angle, AM = $\frac{1}{2}$ AB, and AQ = $\frac{1}{2}$ AD.)

Therefore, MQ is parallel to BD, and MQ = $\frac{1}{2}$ BD.

Simarly, NP is parallel to BD, and NP = $\frac{1}{2}$ BD.

This shows that MQPN is a parallelogram.

Now consider triangle ABO.



If we draw the line XY (as shown), we can show that triangles AMX, MBY, YXM, XYO are all congruent.

For example:

> Triangle MXY is congruent to triangle OYX (Angle, Side, Angle: Angle MXY = Angle OYX, the side XY is common, and Angle XYM = Angle YXO)
>
> Triangle AXM is congruent to triangle MYB (Angle, Side, Angle: Angle XMA = Angle YBM, the side AM = MB, and Angle MAX = Angle BMY)

Therefore, all four triangles have equal area.

Thus the shaded portion of triangle ABO is one half of the triangle.

This argument may be extended to the whole figure.

Shape Statements

1. James says:

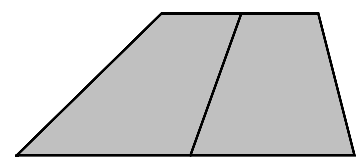
If you draw two shapes, the shape with the greater area will also have the longer perimeter.

Is James' statement Always, Sometimes or Never True?

Fully explain and illustrate your answer.

2. Clara says:

If you join the midpoints of the opposite sides of a trapezoid, you split the trapezoid into two equal areas.



Is Clara's statement Always, Sometimes or Never True?

Fully explain and illustrate your answer.

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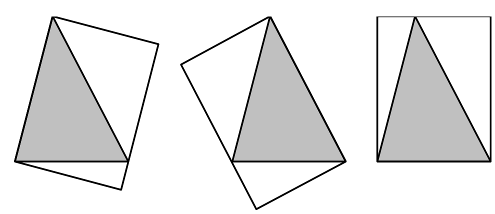


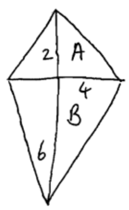
Is Alex's statement Always, Sometimes or Never True?

Fully explain and illustrate your answer.

Student Work: Diagonals of a Quadrilateral

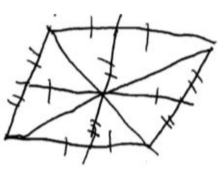
Below is some work by two students. Assess their work by answering these questions:

- What do you like about the work?
- Has the student made any assumptions?
- Is the work accurate?
- How can the explanation be improved?



A	B
$\frac{1}{2} \cdot 4 \cdot 2$	$\frac{1}{2} \cdot 4 \cdot 6$
$\frac{1}{2} \cdot 8$	$\frac{1}{2} \cdot 24$
$\cancel{2}\ 4$	12

$\boxed{\text{Kite = not true}}$

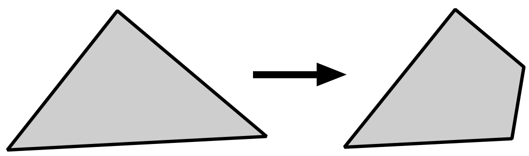


All triangles are congruent (SSS)

$\boxed{\text{Parallelogram = allways true}}$

Card Set A: Always, Sometimes, or Never True?

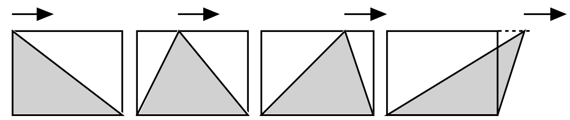
A **Cutting Shapes**



When you cut a piece off a shape you:

(a) Reduce its area.
(b) Reduce its perimeter.

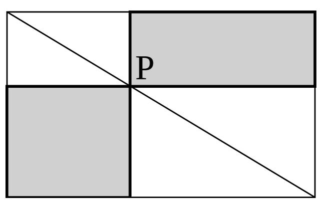
B **Sliding a Triangle**



If you slide the top corner of a triangle from left to right:

(a) Its area stays the same.
(b) Its perimeter changes.

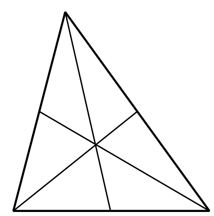
C **Rectangles**



Draw a diagonal of a rectangle and mark any point on it as P. Draw lines through P, parallel to the sides of the rectangle. The two shaded rectangles have:

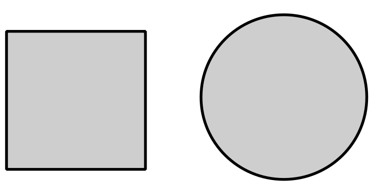
(a) Equal areas.
(b) Equal perimeters.

D **Medians of a Triangle**



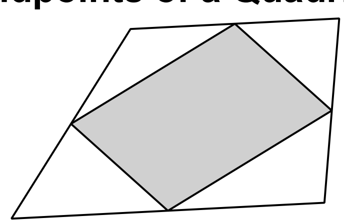
If you join each vertex of a triangle to the midpoint of the opposite side, the six triangles you get all have the same area.

E **Square and Circle**



If a square and a circle have the same perimeter, the circle has the smallest area.

F **Midpoints of a Quadrilateral**



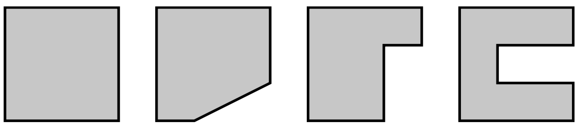
If you join the midpoints of the sides of a quadrilateral, you get a parallelogram with one half the area of the original quadrilateral.

Card Set B: Some Hints

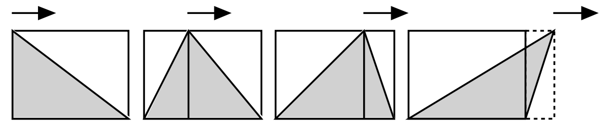
A Cutting Shapes

What happens to the area and perimeter when you cut pieces off this square in different ways?

Generalize!



B Sliding a Triangle

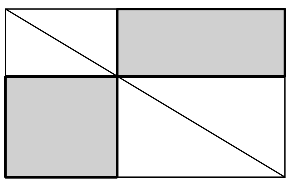


Divide the shaded triangle into smaller triangles as shown.

What fraction of the rectangle is the shaded area in each case?

Will this work in extreme cases?

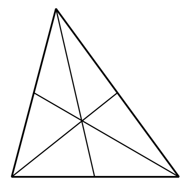
C Rectangles



What happens in extreme cases?
Which pairs of lengths and areas **are** equal?

Don't forget the two large triangles made by the diagonal and the sides of the rectangle.

D Medians of a Triangle



Find pairs of triangles that share the same base and height.

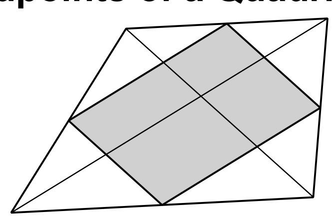
Don't just look at the small triangles!

E Square and Circle

Suppose the square and circle have a perimeter of 4 units.
What are their areas?

Does it matter what the perimeter is? Why?

F Midpoints of a Quadrilateral



Try joining the diagonals of the quadrilateral.

Which lengths and angles must be equal?

Diagonals of a Quadrilateral

If you draw in the two diagonals of a quadrilateral, you divide the quadrilateral into four equal areas.

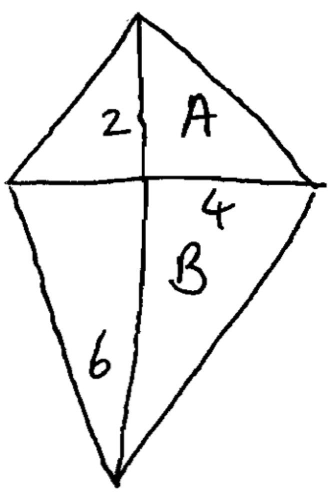
Student Work: Diagonals of a Quadrilateral

- What do you like about the work?
- Has this student made any incorrect assumptions?
- Is the work accurate?
- How can the explanation be improved?

Student Work 1

Diagonals of a Quadrilateral

If you draw in the two diagonals of a quadrilateral, you divide the quadrilateral into four equal areas.



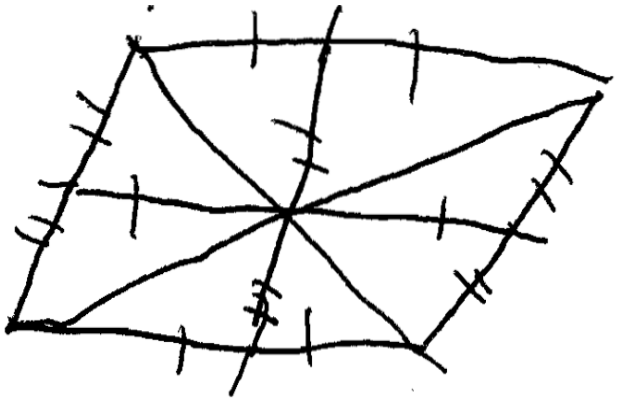
A	B
$\frac{1}{2} \cdot 4 \cdot 2$	$\frac{1}{2} \cdot 4 \cdot 6$
$\frac{1}{2} \cdot 8$	$\frac{1}{2} \cdot 24$
4	12

Kite = not true

Student Work 2

Diagonals of a Quadrilateral

If you draw in the two diagonals of a quadrilateral, you divide the quadrilateral into four equal areas.



All triangles are congruent (SSS)

Parallelogram = allways true

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
Shell Center Team at the Center for Research in Mathematical Education
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with
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org