CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Calculating Volumes of Compound Objects

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Calculating Volumes of Compound Objects

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students solve problems involving measurement and in particular, to identify and help students who have difficulty:

- Computing measurements using formulas.
- Decomposing compound shapes into simpler ones.
- Using right triangles and their properties to solve real-world problems.

STANDARDS ADDRESSED

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

G-SRT: Define trigonometric ratios and solve problems involving right triangles.
N-Q: Reason quantitatively and use units to solve problems.
G-GMD: Explain volume formulas and use them to solve problems.
Visualize relationships between two-dimensional and three-dimensional objects.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practice 3:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.

INTRODUCTION

The unit is structured in the following way:

- Before the lesson, students attempt the task individually. You then review their work and formulate questions for them to answer, in order to improve their work.
- At the start of the lesson, students work individually to answer your questions. Next, they work in small groups on the same task, to produce better collective solutions than those they produced individually. Throughout their work they justify and explain their decisions to each other.
- In the same small groups, students critique some student responses to the *Glasses* task, provided in the *Sample Responses to Discuss*. They correct errors in those responses.
- In a whole-class discussion, students discuss their own solutions and explain how to correct the common misconceptions seen in the *Sample Responses to Discuss*.
- Finally, students work on their own again to improve their individual solutions.

MATERIALS REQUIRED

- Each student will need two copies of the *Glasses* task, the *Formula Sheet*, a mini-whiteboard, pen, and eraser.
- Each small group of students will need a new copy of the *Glasses* task and the *Sample Responses to Discuss*.
- Some students may require the *Glasses: Extension Questions*. Have copies of the *Hint Sheet* ready in case any students need it.

TIME NEEDED

15 minutes before the lesson, a 1-hour lesson, and 15 minutes in a follow-up lesson. All timings are approximate and will depend on the needs of the students.

BEFORE THE LESSON

Assessment task: *Glasses* (15 minutes)

Have students do this task, in class or for homework, a day or more before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.

Give each student a copy of the *Glasses* task.

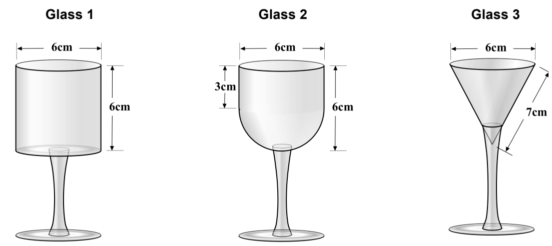
> *Spend 15 minutes working individually to answer these questions.*
>
> *Show all your work on the worksheet.*

It is important that, as far as possible, students answer the questions without assistance.

Students who sit together often produce similar answers and then when they come to compare their work, they have little to discuss. For this reason, we suggest that, if students do the assessment task in class, you ask them to move to different seats. Then at the beginning of the formative assessment lesson, allow them to return to their usual places. Experience has shown that this produces more profitable discussions.

Students should not worry too much if they cannot understand or do everything, because there will be a lesson using this task, which should help them. Explain to students that by the end of that lesson, they should expect to answer questions such as these with confidence. This is their goal.

Glasses



This picture shows three glasses.

The measurements are all in centimeters.
The bowl of Glass 2 has a cylindrical top and a hemispherical base.
The bowl of Glass 3 goes down into the stem.

1. Calculate the volume of liquid that would fill the bowl of each glass.
   Show all your calculations and explain your reasoning.

a. Glass1

....

....

....

b. Glass 2

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....

....

c. Glass 3

2. Find the height of liquid in Glass 2 when it is half full.
   Show how you figure it out.

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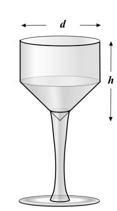
3. Glass 4 is shown in the diagram.
   One of the following formulae gives the volume of Glass 4.
   Which is it?

$\frac{1}{6}\pi d^2 h$ $\quad$ $\frac{1}{6}\pi dh$ $\quad$ $\frac{1}{6}\pi dh^2$ $\quad$ $\frac{1}{6}\pi d^2 h^2$



Glass 4

Explain how you can tell by just looking at the form of these expressions.

....

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding. The purpose of this is to forewarn you of issues that will arise during the lesson itself, so that you may prepare carefully.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given in the *Common issues* table on page T-4. These have been drawn from common difficulties observed in trials of this unit.

We suggest that you make a list of your own questions, based on your students' work, using the ideas on the following page. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions relevant to each student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work at the beginning of the lesson.

Common issues	Suggested questions and prompts
Has difficulty in identifying the values to substitute for variables in the formula For example: The student does not match the variables in the formula with measures on the figure when applying a formula, using diameter rather than radius, or multiplying arbitrary numbers (Q1). Or: The student uses slant height rather than the Pythagorean Theorem to find perpendicular height (Q1c).	• What measures do the variables in your formula stand for? Can you draw these in on the diagram?
Makes calculation errors For example: The student makes a numerical error in calculation such as doubling rather than squaring.	• How can you check your answers?
Chooses wrong formula For example: The student chooses the pyramid formula for a prism (Q1).	• What is the difference between a prism and a pyramid? • Is a cylinder a prism or a pyramid? Explain your answer.
Has difficulty decomposing a 3D shape For example: The student does not calculate the volume of a compound shape using appropriate formulas for constituent parts. There is no attempt, use of a single incorrect formula, or an incorrect decomposition.	• Imagine you can take this 3D shape to pieces. What pieces would you make in order to calculate the volume using formulas you know? • For Glass 3, the bowl of the glass goes down into the stem.
Assumes proportionality For example: The student assumes that halving the volume also halves the height, giving a response of 3 cm (Q2).	• Look back at Q1b. Are the volumes of the two parts of the figure the same? How does this affect your answer?
Does not discriminate between length, area, and volume formulas For example: The student multiplies too many or too few numbers together to calculate a volume. Or: The student chooses an incorrect formula that involves the square of the height (Q3).	• What is the difference between a length, an area, and a volume formula? • Compare the formulas for the cylinder and cone. What is their common base area? How does that show in the formulas?
Answers all questions correctly The student needs an extension task.	• Write down how someone could decide which formulas represent volumes. • Find the height of the liquid in Glass 3 when it is half full. Explain your answer. • Show step-by-step how the formula given in question 4 was derived. • Make up a challenging glass volume problem of your own and solve it.

SUGGESTED LESSON OUTLINE

Improve individual solutions to the *Glasses* task (10 minutes)

Give each student a mini-whiteboard, pen, and eraser.

Recall what we were working on previously. What was the task?

Return students' written answers to the *Glasses* problem. If you have not added questions to students' work, write a short list of your most common questions on the board. Students can then select a few questions appropriate to their own work and begin answering them.

Ask students to re-read the *Glasses* task and their own work. Draw their attention to the questions you have about their work.

I read your solutions and have some questions about your work.

I would like you to work on your own to answer my questions.

Write your answers on your mini-whiteboard, so I can see what you're thinking about as I walk around.

The aim of this activity is to encourage students to re-engage with their work and reflect on what they have done. Often students comment that they find even their own reasoning hard to follow! This is a sure sign that they need to work on the quality of their explanations.

Collaborative small-group work on the *Glasses* task (20 minutes)

Organize the students into small groups of two or three and give each group of students a new copy of the *Glasses* task.

Now I would like you to work in small groups.

Take turns to explain what you did on the problems and where you got stuck.

Then, together, try to produce a solution that is better than your individual solutions.

As students work, you have two tasks: to note students' approaches to the task and to support student problem solving.

Note student approaches to the task

Listen to students and identify the issues they are discussing. In particular, listen to see whether they are addressing the difficulties outlined in the *Common issues* table. Which parts of the task do they struggle with most? You can use this information to focus the whole-class discussion later in the lesson.

Support student problem solving

From time to time, intervene and ask questions to help students clarify their thinking. Try not to help students too much by 'taking over' their work. Instead, encourage students to explain to one another. If several students in the class are struggling with the same issue, write a relevant question on the board.

If a group of students continues to struggle with identifying the missing lengths and constructions in Q2 and Q3 of *Glasses*, offer them a copy of the *Hint Sheet*.

For students who are succeeding with the task, suggest that they try to find the height of liquid in Glass 3 when it is half full. This is a more demanding problem.

Collaborative analysis of *Sample Responses to Discuss* (15 minutes)

Give each small group of students a copy of each of the *Sample Responses to Discuss*.

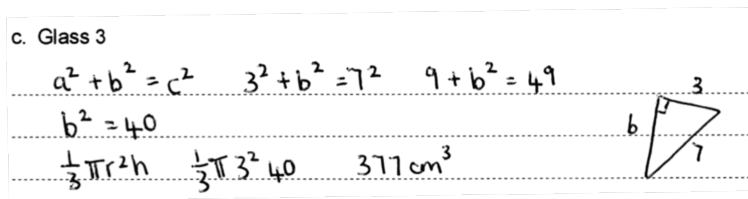
None of the sample responses shows the correct answer for the volume.

Work together to find and explain the errors each student made.

Explain what the student could do to complete his or her solution correctly.

The sample responses show some of the common errors made by students on Q1c and Q2 in trials. This task will give students an opportunity to address those common misconceptions, without providing a complete solution strategy.

Logan uses the correct formula for the volume of a cone. He identifies that the correct height to substitute into that formula is the perpendicular height of the cone. He uses the Pythagorean Theorem to figure out the perpendicular height but forgets to find the square root of $b^2 = 40$. This leads to an incorrect solution. Logan should recalculate the cone volume using $2\sqrt{10}$ for the perpendicular height.

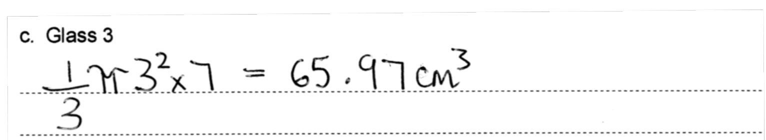

c. Glass 3

$a^2 + b^2 = c^2$ $3^2 + b^2 = 7^2$ $9 + b^2 = 49$

$b^2 = 40$

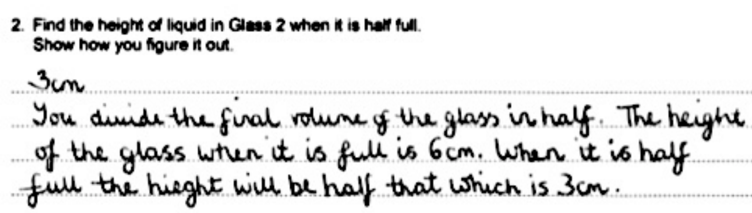
$\frac{1}{3}\pi r^2 h$ $\frac{1}{3}\pi 3^2\, 40$ $377\ cm^3$

Isaac uses the correct formula for the volume of a cone but uses an incorrect measure. He uses the slant height rather than the perpendicular height. He should first calculate the perpendicular height of the cone using the Pythagorean Theorem and then substitute that value into the formula.


c. Glass 3

$\frac{1}{3}\pi 3^2 \times 7 = 65.97\ cm^3$

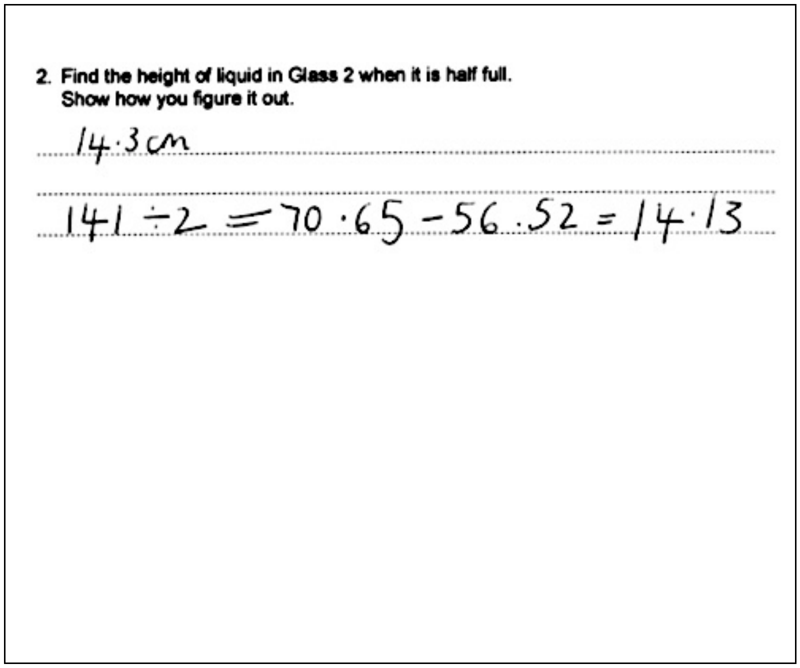
Yasmin has assumed that halving the volume halves the height of the liquid in the glass. This is a false assumption. To figure out the correct solution, Yasmin should first halve the total volume of the glass and then subtract the volume of the hemisphere from the half volume. She will then know the volume of liquid in the cylinder. She can calculate the height of the liquid and add that to the height of liquid in the hemisphere.


2. Find the height of liquid in Glass 2 when it is half full.
Show how you figure it out.

3cm

You divide the final volume of the glass in half. The height of the glass when it is full is 6cm. When it is half full the hieght will be half that which is 3cm.

Brianna calculates half the volume of the glass using 3.14 for π. She has a total volume of 141.3, but has written 141. She then subtracts the volume of the hemisphere. This shows how much of the volume of the cylinder is taken up when the glass is half full. Brianna needs to complete her solution by working out the height of a cylinder of radius 3 cm that has this volume. That height, added to the height of the hemisphere (3 cm), is the height of liquid in the half-full glass. Brianna also uses the equals sign in a non-standard way.


2. Find the height of liquid in Glass 2 when it is half full.
Show how you figure it out.

14·3 cm

141 ÷ 2 = 70·65 − 56.52 = 14·13

Whole-class discussion of *Sample Responses to Discuss* (15 minutes)

Organize a whole-class discussion. The intention is for you to focus on getting students to understand the different methods of working out the answers, to articulate the common misconceptions and to share their reasoning, rather than to check numerical solutions.

Let's stop and talk about the different errors the students made.

Look at Logan and Isaac's solutions to Question 1c.

Which student produced the better solution? Explain your response.

Which height do you need to calculate this volume?

Zsa-Zsa, your group wrote an explanation of that. Could you explain for us please?

Put Logan's/Isaac's explanation into your own words.

Look at Yasmin and Brianna's solutions to Question 2.

Which student produced the better solution? Explain your response.

Follow-up lesson: review individual solutions to the assessment task (15 minutes)

Return students' individual papers from the *Glasses* assessment task along with a fresh copy of the task sheet.

Read through your original responses and think about what you have learned.

Using what you have learned, try to improve your solution.

If a student is satisfied that they have completed the task satisfactorily, ask them to try some of the *Glasses: Extension Questions*. Question 3 on this sheet is much more difficult than the previous questions, so it should provide an appropriate challenge for students who have succeeded on the other parts of this task.

Some teachers ask students to work through this task for homework.

SOLUTIONS

Assessment task: *Glasses*

1a) The volume of Glass 1 = $\pi \times 3 \times 3 \times 6 = 54\pi = 170$ cm^3.

1b) The volume of the hemisphere = $(4\pi \times 3^3) \div 6 = 18\pi = 56.5$.
The volume of the cylinder = $\pi \times 3^2 \times 3 = 27\pi = 84.8$.

Total volume of Glass 2 = $45\pi = 141$ cm^3.

1c) Using the Pythagorean Theorem, the height of the cone is $\sqrt{7^2 - 3^2} = \sqrt{40} = 2\sqrt{10}$.

The volume of Glass 3 $= (\pi \text{ x } 3^2 \text{ x } 2\sqrt{10}) \div 3 = 6\sqrt{10}\pi = 60$ cm^3.

2. The volume of liquid in the half-full Glass 2 is $141 \div 2 = 70.5$.

The volume of liquid in the cylinder $= 70.5 - 56.5 = 14$.

$14 = \pi \times 3^2 \times$ height in cylinder.

Height in cylinder $= 14 \div 9\pi = 0.5$.

The total height $= 3 + 0.5 = 3.5$ cm.

3. Glass 4 is composed from a cylinder and cone. While we do not yet have enough information to deduce the formula, it is possible to rule out three of the formulas on grounds that the dimensions of the formulas are incorrect:

$\frac{1}{6}\pi dh$ Only two lengths are multiplied so this has the dimension of an area.

$\frac{1}{6}\pi d^2 h^2$ Four lengths are multiplied so this is not a volume either.

Both $\frac{1}{6}\pi d^2 h$ and $\frac{1}{6}\pi dh^2$ involve multiplying together three lengths but $\frac{1}{6}\pi dh^2$ involves the square of the height and so cannot be correct.

The correct formula is therefore $\frac{1}{6}\pi d^2 h$.

Glasses: Extension Questions

1. This question is intended to encourage the discussion of dimensional analysis.

When lengths are combined by addition we obtain another length.
If two lengths are multiplied we obtain an area.
If three are multiplied we obtain a volume.

2. The volume of the Glass = volume of cylinder + volume of cone

$$= \pi\left(\frac{d}{2}\right)^2\left(\frac{h}{2}\right) + \frac{1}{3}\pi\left(\frac{d}{2}\right)^2\left(\frac{h}{2}\right)$$

$$= \frac{4}{3}\pi\left(\frac{d}{2}\right)^2\left(\frac{h}{2}\right)$$

$= \frac{1}{6}\pi d^2 h$.

3. When Glass 3 is half full, it will hold 30 cm^3 (from Q1c).

If the height of liquid is h and the radius of the top of the liquid is r then $\frac{1}{3}\pi r^2 h = 30$.

So $\pi r^2 h = 90$. (1)

By similar triangles:

The ratio *height of bowl : radius of bowl* = $h : r = 2\sqrt{10} : 3$

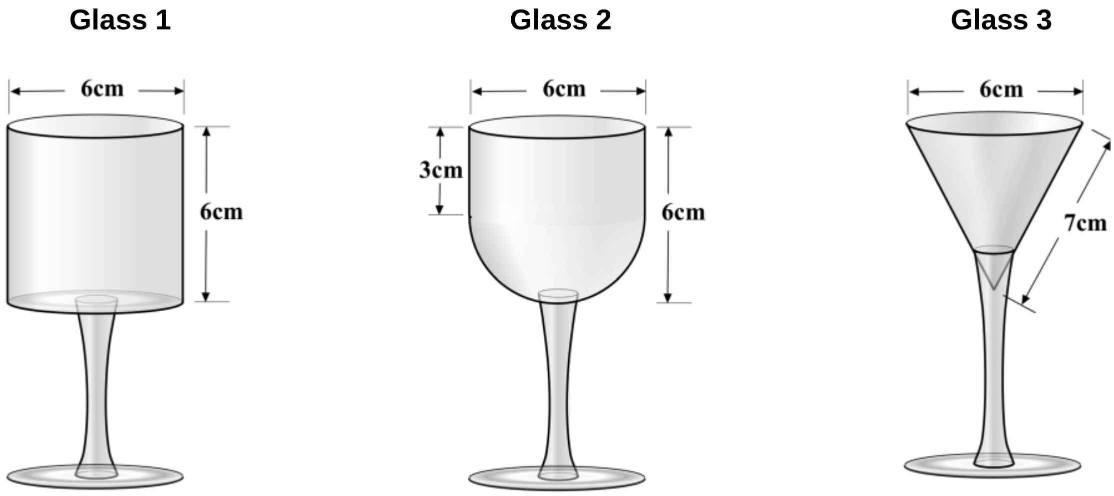
This means that $r = 0.47h$ (2)

Substituting (2) in (1):

$0.225\, h^3 = 28.64 \ldots$, and so $h = 5.03 \ldots$ cm.

The height of liquid will be 5.0 cm (to 1 decimal place).

Glasses



This picture shows three glasses.

The measurements are all in centimeters.
The bowl of Glass 2 has a cylindrical top and a hemispherical base.
The bowl of Glass 3 goes down into the stem.

1. Calculate the volume of liquid that would fill the bowl of each glass.
   Show all your calculations and explain your reasoning.

a. Glass 1

----

b. Glass 2

----

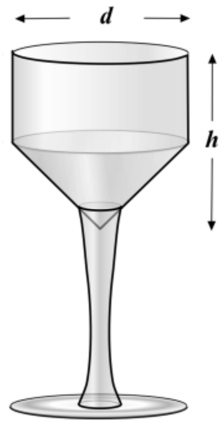
c. Glass 3

----

2. Find the height of liquid in Glass 2 when it is half full.
Show how you figure it out.

3. Glass 4 is shown in the diagram.
One of the following formulae gives the volume of Glass 4.
Which is it?

$\frac{1}{6}\pi d^2 h$ $\frac{1}{6}\pi dh$ $\frac{1}{6}\pi dh^2$ $\frac{1}{6}\pi d^2 h^2$



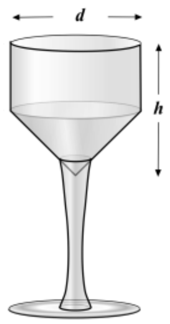
Glass 4

Explain how you can tell by just looking at the form of these expressions.

Glasses: Extension Questions

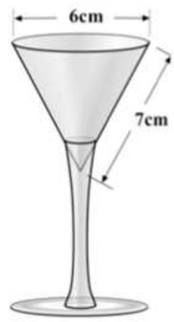
1. Explain how you can tell when a formula represents a length, an area, or a volume.

2. Show step by step how a formula for the volume of Glass 4 may be derived.



Glass 4

3. Find the height of liquid in Glass 3 when it is half full.
   Explain your answer.

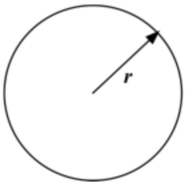


Glass 3

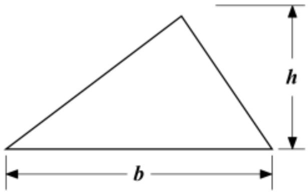
4. On a separate sheet of paper, make up your own Glass question and solve it.
   Try to make your problem challenging, but solvable!
   Now give it to someone else in your class to solve.

Formula Sheet

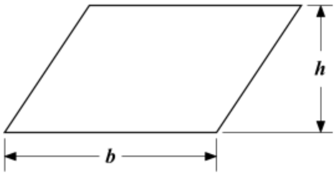
Area of a circle: πr^2



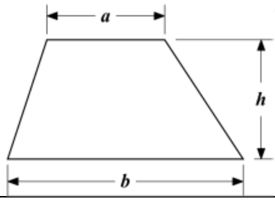
Area of a triangle: $\frac{bh}{2}$



Area of a parallelogram: bh

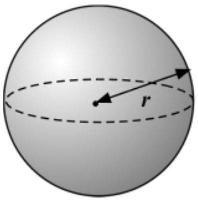


Area of a trapezoid: $\frac{1}{2}(a+b)h$

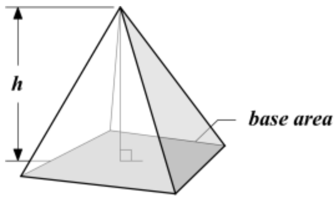


Surface area of a sphere: $4\pi r^2$

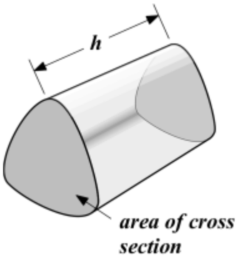
Volume of a sphere: $\frac{4}{3}\pi r^3$



Volume of a pyramid: $\frac{1}{3}$ *base area* $\times h$

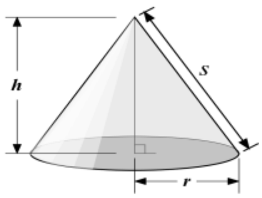


Volume of a prism: $\text{area of cross section} \times \text{height}$



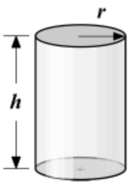
Volume of a cone: $\frac{1}{3}\pi r^2 h$

Curved surface area of cone: πrs



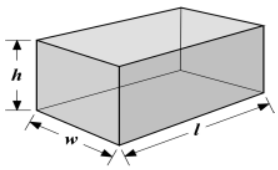
Volume of a cylinder: $\pi r^2 h$

Curved surface area of cylinder: $2\pi rh$

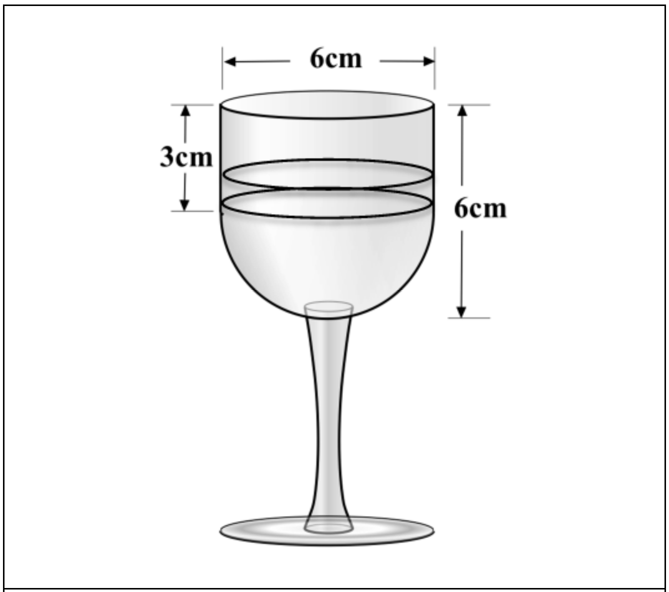


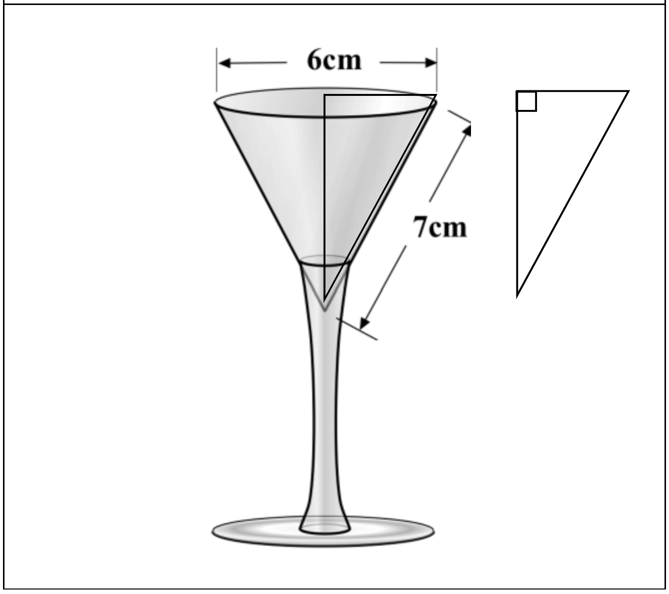
Volume of a rectangular prism: lwh

Surface area of rectangular prism: $2(wh+lh+wl)$



Hint Sheet





Sample Responses to Discuss

Here is some work on the *Glasses* task done by other students.

Neither Isaac nor Logan has found the correct volume for Glass 3.

Neither Brianna nor Yasmin has found the correct height of liquid in Glass 2 when it is half full.

1. Find and explain the errors each student made.
2. Explain what the student needs to do to complete his or her solution correctly.

For example, you might write sentences beginning like these ones:

The student has substituted an incorrect measure …

This is the wrong measure because ...

The correct measure is …

The correct solution is …

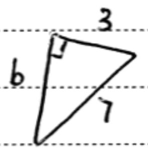
Logan's solution

c. Glass 3

$a^2 + b^2 = c^2$ $3^2 + b^2 = 7^2$ $9 + b^2 = 49$

$b^2 = 40$

$\frac{1}{3}\pi r^2 h$ $\frac{1}{3}\pi 3^2 40$ $377 cm^3$



Isaac's solution

c. Glass 3

$\frac{1}{3}\pi 3^2 \times 7 = 65.97 cm^3$

Yasmin's solution

2. Find the height of liquid in Glass 2 when it is half full.
Show how you figure it out.

3cm

You divide the final volume of the glass in half. The height of the glass when it is full is 6cm. When it is half full the hieght will be half that which is 3cm.

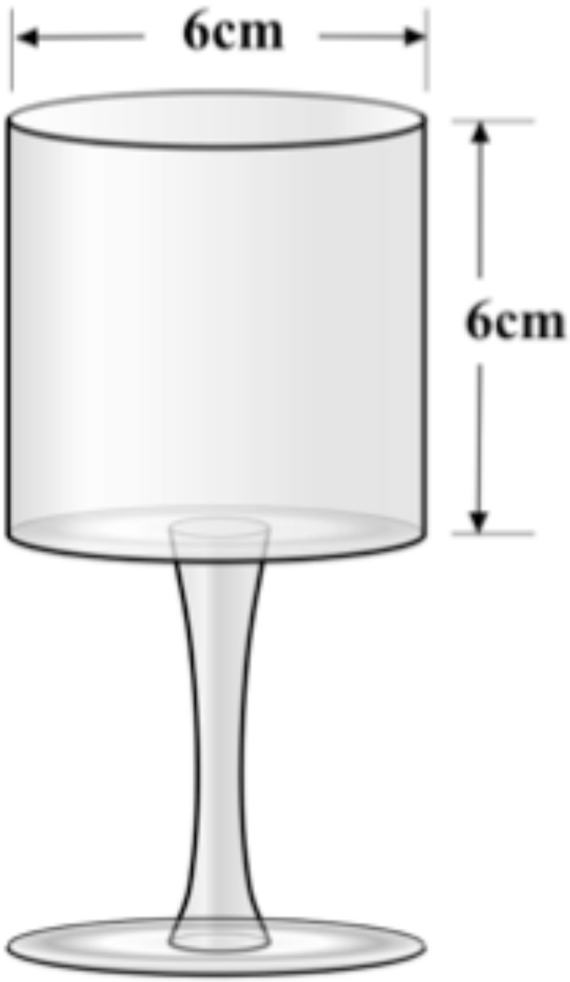
Brianna's solution

2. Find the height of liquid in Glass 2 when it is half full.
Show how you figure it out.

14.3 cm

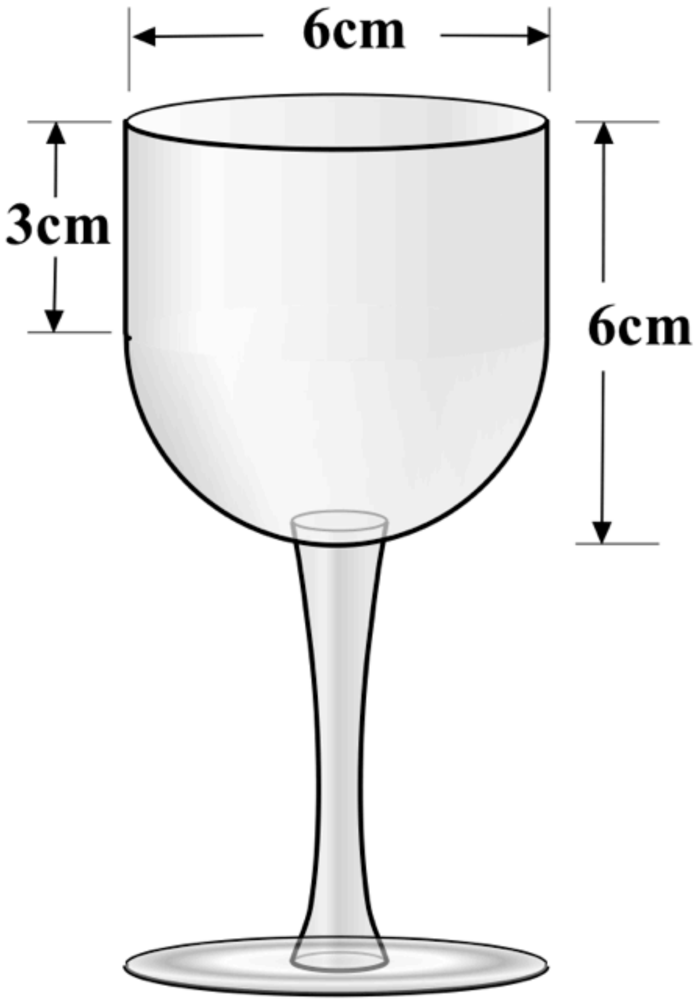
141 ÷ 2 = 70.65 − 56.52 = 14.13

Glass 1



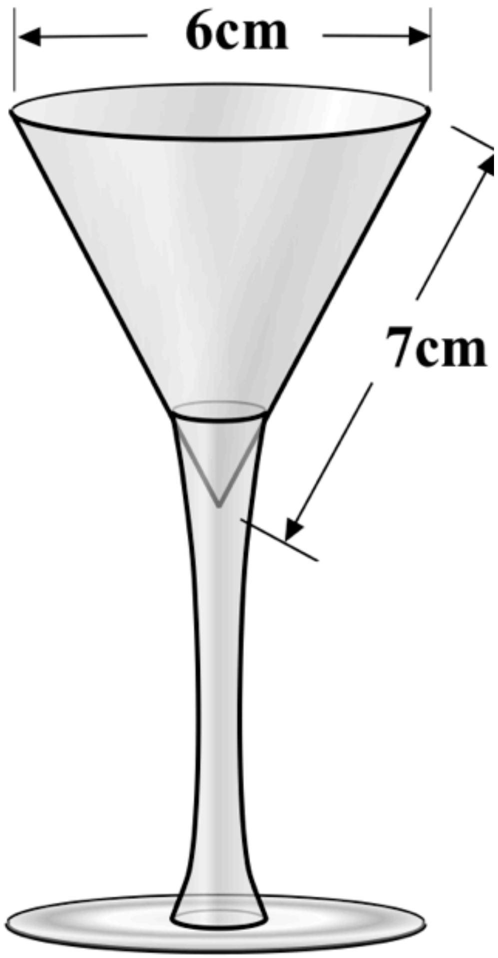
Calculate the volume of liquid that would fill the bowl of the glass.

Glass 2



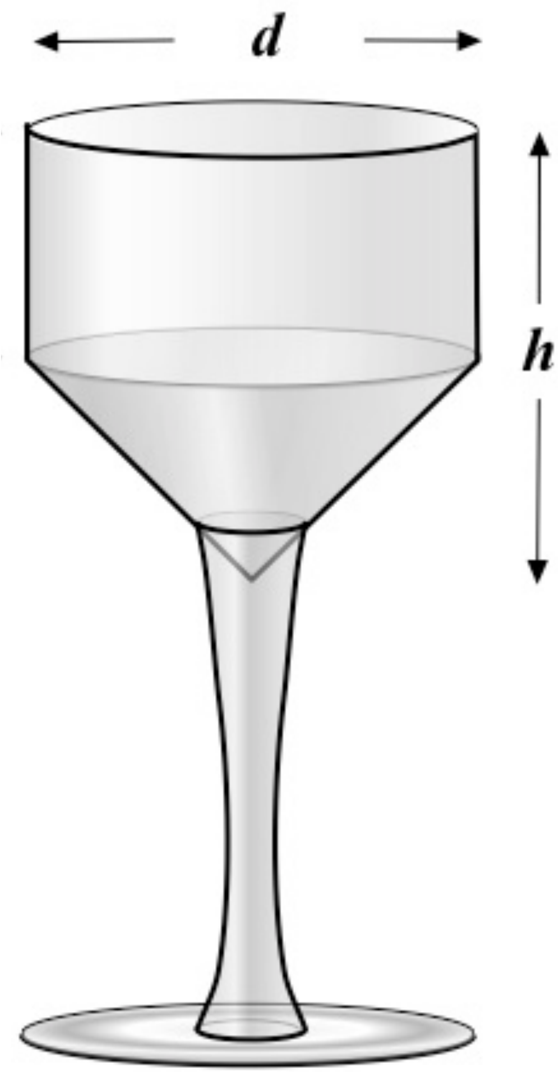
Calculate the volume of liquid that would fill the bowl of the glass.

Glass 3



Calculate the volume of liquid that would fill the bowl of the glass.

Glass 4: Which formula gives the volume?



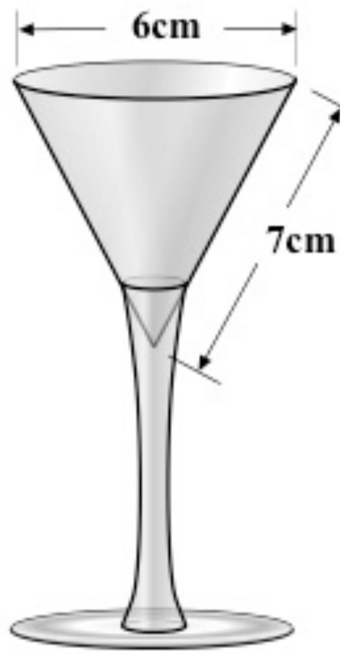
$$\frac{1}{6}\pi dh$$

$$\frac{1}{6}\pi d^2h^2$$

$$\frac{1}{6}\pi d^2h$$

$$\frac{1}{6}\pi dh^2$$

Sample Responses to Discuss: Logan

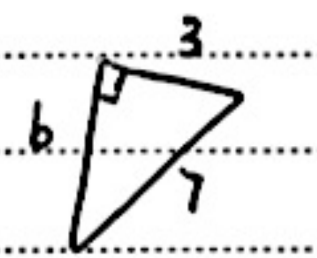


c. Glass 3

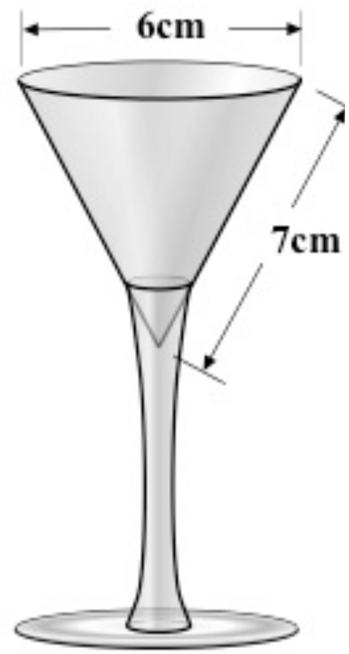
$a^2 + b^2 = c^2$ $\quad 3^2 + b^2 = 7^2$ $\quad 9 + b^2 = 49$

$b^2 = 40$

$\frac{1}{3}\pi r^2 h$ $\quad \frac{1}{3}\pi 3^2 40$ $\quad 377\text{ cm}^3$



Sample Responses to Discuss: Isaac

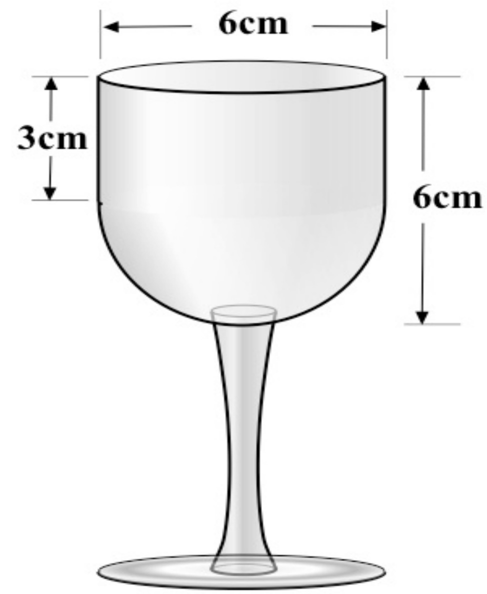


c. Glass 3

$\frac{1}{3}\pi 3^2 \times 7 = 65.97 cm^3$

Sample Responses to Discuss: Yasmin

Find the height of liquid when it is half full.

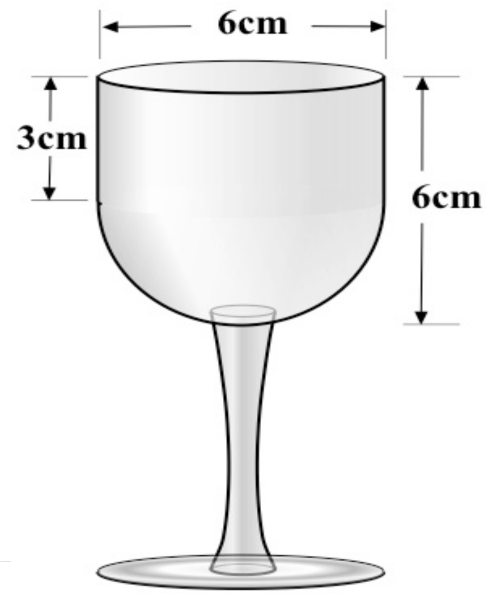


3cm

You divide the final volume of the glass in half. The height of the glass when it is full is 6cm. When it is half full the hieght will be half that which is 3cm.

Sample Responses to Discuss: Brianna

Find the height of liquid when it is half full.



14·3cm

141 ÷ 2 = 70·65 − 56.52 = 14·13

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
Shell Center Team at the Center for Research in Mathematical Education
University of Nottingham, England:

Malcolm Swan,
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with
Hugh Burkhardt, Rita Crust, Andy Noyes, and **Daniel Pead**

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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org