CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Sorting Equations of Circles 1

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Sorting Equations of Circles 1

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Use the Pythagorean theorem to derive the equation of a circle.
- Translate between the geometric features of circles and their equations.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

G-GPE: Translate between the geometric description and the equation for a conic section.
A-CED: Create equations that describe numbers or relationships.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 2, 6, 7, and 8:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

- Before the lesson, students work individually on an assessment task that is designed to reveal their current understanding and difficulties. You then review their work and create questions for students to answer in order to improve their solutions.
- After a whole-class introduction students work collaboratively, categorizing equations and geometric descriptions of circles.
- At the end of the lesson there is a whole-class discussion.
- In a follow-up lesson, students again work alone on a similar task to the pre-lesson task.

MATERIALS REQUIRED

- Each student will need a copy of the assessment tasks, *Going Round in Circles* and *Going Round in Circles (revisited)*, a mini-whiteboard, pen, and eraser.
- Each small group of students will need *Card Set: Equations* (cut up before the lesson), an enlarged copy of the *Categorizing Equations* table, a compass, and a glue stick.
- You may want to copy the *Card Set: Equations* and the *Categorizing Equations* table onto transparencies to be used on an overhead projector to support whole-class discussions.
- There is a projector resource to support whole-class discussions.

TIME NEEDED

15 minutes before the lesson, a 70-minute lesson (or two shorter lessons), and 15 minutes in a follow-up lesson. Timings are only approximate.

BEFORE THE LESSON

Assessment task: *Going Round in Circles* (15 minutes)

Give this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the next lesson.

Give each student a copy of the assessment task *Going Round in Circles*.

Read through the questions and try to answer them as carefully as you can.

It is important that, as far as possible students are allowed to answer the questions without your assistance.

Students should not worry too much if they cannot understand or do everything, because in the next lesson they will engage in a similar task, which should help them. Explain to students that by the end of the next lesson they should expect to be able to answer questions such as these confidently. This is their goal.

Going Round in Circles

You may want to use the space to the right of the questions to sketch graphs.

1. The end points of the diameter of a circle are (6, 0) and (−6, 0).

   a. What are the coordinates of the center of the circle?

   b. What is the equation of *this* circle?
   Fully explain your answer.

   c. A point on *this* circle has coordinates (2, *m*). Write possible values for *m*.
   Fully explain your answer.

2. The center of another circle is (−5, 1). Its radius is $\sqrt{14}$
   What is the equation of this circle?
   Fully explain your answer.

Assessing students' responses

Collect students' responses to the task and note what their work reveals about their current levels of understanding and their different approaches.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this unit. We suggest you make a list of questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these questions on the board when you return the work to the students in the follow-up lesson.

Common issues:	**Suggested questions and prompts:**
Incorrectly identifies the center of the circle For example: The student writes (6, 6) (Q1a).	• On a coordinate grid mark the two end points of the diameter, then sketch a circle through these two points. What are the coordinates of the center?
Incorrectly identifies the equation of the circle or does not answer the question For example: The student writes $x^2 + y^2 = 6$ (Q1b).	• What is the question asking you to find? • What do you already know about the circle? • How can you present this information? • How can you use this information to check your equation is correct? • It may help if you sketched the circle. • Use math to figure out the coordinates of the point (x, y) on the graph. Describe your method.
Sketches the graph of the circle and uses it to approximate the coordinates of a point on the circumference (Q1c)	• What math can you use to check that the coordinates are accurate?
Correctly figures out one value for *m* (Q1c)	• Can you now figure out a second possible value for *m*?
Incorrectly uses the Pythagorean theorem to figure out the equation of the second circle For example: The student writes $(x - 5)^2 + (y + 1)^2 = 14$ (Q2).	• How can you check your equation is correct? • Sketch the circle.
Provides little explanation in answers	• Add more explanation so that someone unfamiliar with the math can understand your answer.
All answers are correct The student needs an extension task.	• Figure out the coordinates of any *x*-intercepts or *y*-intercepts of this circle. • The equation of a circle is $(x - 4)^2 + (y + 2)^2 = p$. Where *p* is an integer. Figure out a value for *p*, that ensures the circle crosses the *x*-axis but not the *y*-axis.

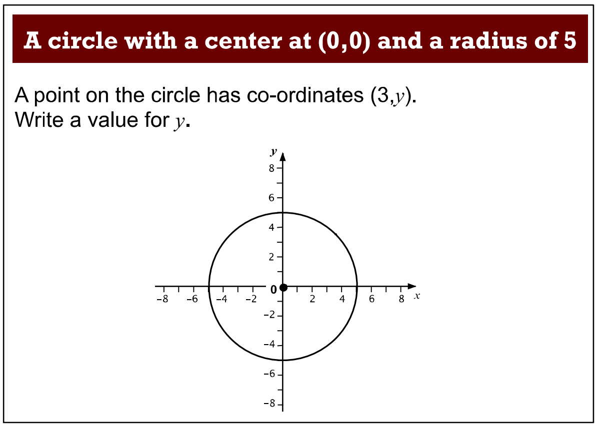
SUGGESTED LESSON OUTLINE

Whole-class introduction (15 minutes)

Give each student a mini-whiteboard, pen and eraser.

Throughout the introduction, encourage students to first tackle a problem individually and only then discuss it with a neighbor. In that way students will have something to talk about. Maximize participation in the whole-class discussion by asking all students to show you solutions on their mini-whiteboards. Select a few students with interesting or contrasting answers to justify them to the class. Encourage the rest of the class to challenge these explanations.

Show Slide P-1 of the projector resource:



Introduce the task with:

Can you find the co-ordinates of any point on this circle? [e.g. (5, 0); (-5, 0); (0, 5); (0, -5).]

Then ask students to use their mini-whiteboards to respond to the following question:

A point on the circle has coordinates (3, y).
Write a value for y. *[y = 4 or y = −4.]*

If you find students are estimating the value of *y*, encourage them to use math to check the accuracy of their answer. After a few minutes ask a couple of students with different values for *y* to justify their answers.

If students struggle to use math to calculate an accurate answer, then you may choose to ask the following questions:

Does it help if I draw some additional lines on the grid? [E.g. From the origin to (3, y) and a vertical line from (3, y) to the x-axis.]

How can you use the Pythagorean theorem to figure out the value for y?

Then ask the following questions:

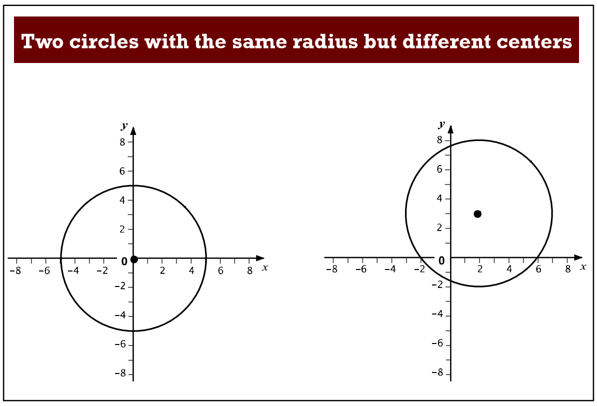
Can you think of another point on the circle, for which both coordinates are integers? Write all the points you can think of. [E.g. (3, 4); (4, 3); (−4, −3).]

Now give me the coordinates of a point on the circle for which the x *value is an integer, but the* y *value is not an integer. Give me the exact value! You can leave your coordinates in square root form. [E.g.* $(1,\sqrt{24})$; $(-1,\sqrt{24})$; $(2,\sqrt{21})$; $(2,-\sqrt{21})$ *etc.]*

Now give me the coordinates of a different point on the circle, for which the y *value is an integer, but the* x *value is not an integer.*

Mark a point (x, y) on the circle and ask students to figure out the equation satisfied by this point. After a few minutes ask a few students to justify their answers. [$x^2 + y^2 = 5$.]

Show Slide P-2 of the projector resource:



Carefully introduce the second stage:

We are now going to move the circle so that its center is at (2, 3). Its radius remains the same.

Figure out the coordinates of two points on this new circle. Select one point that is easy to figure out and the other difficult.

Explain why your points are easy and difficult to figure out.

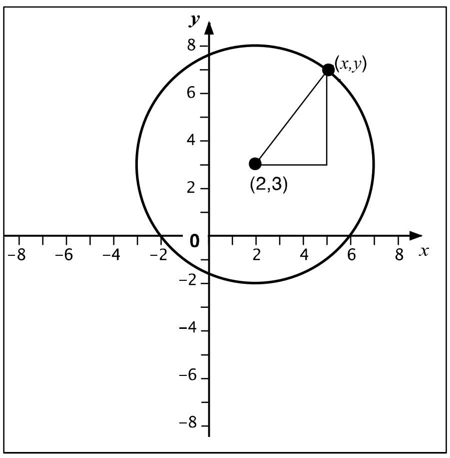
Students could decide the easy points to find are points on the x-axis or the end points of the horizontal diameter or the vertical diameter: (6, 0), (-2, 0), (7, 3), (2, 8) etc.

Again ask two or three students to explain their answers. You may find students use the Pythagorean theorem or they may translate the coordinates of a point on the first circle +2 horizontally and +3 vertically.

Finally ask:

What is the equation of the circle? [$(x-2)^2 + (y-3)^2 = 25$.]

How can you see this is true using the Pythagorean theorem?



For classes that may struggle with the introduction

You may decide to first show students a circle drawn on a grid, marked with the coordinates of its center and the coordinates of a point on the circumference. Ask students to figure out the radius of the circle. This should encourage students to use the Pythagorean theorem.

Collaborative activity: making posters (25 minutes)

Organize the class into small groups of two or three students.

Give each group of students a copy of *Card Set: Equations* and an enlarged copy of *Categorizing Equations*.

> *You are now going to continue to explore linking the equation of a circle with its center and radius.*
>
> *On your desk you should have twelve equation cards and one table: Categorizing Equations.*

Explain how students are to work collaboratively. Slide P-3 of the projector resource summarizes these instructions:

Categorizing Equations

1. Take turns to place an equation card in one of the categories in the table.
2. If you place a card, explain how you came to your decision.
3. If you don't agree or understand, ask your partner to explain their reasoning.
4. Write additional information or include a drawing as part of your explanation.
5. Some of your cards are to go in one of the cells in the final column.
   – You will need to figure out the coordinates for the center of the circle for all equations placed in this column.
6. Make up your own equation for any empty cells.

You all need to agree on and explain the placement of every card.

Students may not have time to place all the equation cards. It is better if they explain their reasoning fully for a few cards than rush through trying to place all the cards.

The purpose of this structured work is to encourage students to engage with each other's explanations and take responsibility for each other's understanding.

While students work in small groups you have two tasks: to note different student approaches to the task and to support student reasoning.

Note different student approaches to the task

Notice how students make a start on the task, any interesting ways of explaining a categorization, any attempts to generalize, where they get stuck, and how they respond if they do come to a halt. For example, do students plot the graph of an equation accurately? Do students use the Pythagorean theorem to figure out the coordinates of the point (x, y) on the circle? Do students use guess and check? Do students notice that the equations in one row or column have a common feature? You can use this information to focus a whole-class discussion at the end of the lesson.

Support student reasoning

Try not to make suggestions that move students towards a particular categorization. Instead, ask questions to help students to reason together. You may choose to use some of the questions and prompts from the *Common issues* table.

If students struggle to get started, encourage them to ask you a specific question about the task. Articulating the problem in this way can sometimes offer a direction to pursue that was previously overlooked. You could then ask another member of the group for a response.

You may want to follow this with:

> *For this cell, what do you know about the circle? How can you present this information?*
> *What do you need to know in order to place an equation card in this cell?*

Some students may try to accurately plot the graph of the equation. Encourage them to sketch the circle instead.

> *Is it easier to sketch the circle using the equation or using the information for one category of the table? Why is it easier?*
>
> *What math can you now use to figure out the coordinates of a point (x, y) for a circle to go in this cell?*

Students may use the information in the table to sketch a circle and then substitute the coordinates of a point on this circle (for example, the coordinates of the end point of a horizontal diameter) into the left side of an equation. Next they might check to see if this value equates with the right side of the equation. Encourage students to reflect on this way of working and to investigate a different method.

Once students have correctly placed two or three cards along the same row or column of the table, encourage them to look at the structure of the equation and start to make generalizations.

> *Do the equations you've placed in this row/column have anything in common?*
>
> *What does this equation tell you about its graph?*

If you find one student has placed a card in a particular category on the table, challenge another student in the group to provide an explanation.

> *Chan placed this card. Cheryl, why does Chan think this equation goes here?*

If you find the student is unable to answer this question, ask them to discuss the work further. Explain that you will return in a few minutes to ask a similar question.

Sharing posters (15 minutes)

As students finish matching the cards, ask one student from each group to visit another group's poster.

You may want to use Slide P-4 of the projector resource to explain what students should do:

Sharing Posters

1. If you are staying at your desk, be ready to explain the reasons for your group's placement of equations in the table.

2. If you are visiting another group, copy your table onto a piece of paper. Only write the equation numbers in each cell of the table, not the whole equation.
   - Go to another group's desk and check to see which cells are different from your own.
   - If there are differences, ask for an explanation. If you still don't agree, explain your own thinking.
   - When you return to your own desk you need to consider as a group whether to make any changes to your own table.

Give each small group a glue stick. When students are satisfied, they should glue the equations onto the poster.

Whole-class discussion (15 minutes)

Organize a whole-class discussion about different strategies used to place the cards. The intention is that you work with the students to make the math of the lesson more explicit and encourage students to make generalizations.

First select a card that most groups placed correctly. This approach may encourage good explanations. Then select one or two cards that most groups found difficult to place.

Once one group has justified their choice for a particular placement, ask other students to contribute ideas of alternative approaches and their views on which reasoning method was easier to follow. The intention is that you focus on getting students to understand and share their **reasoning**, not just checking that everyone produced the right answers. Use your knowledge of the students' individual and group work to call on a wide range of students for contributions. You may want to draw on the questions in the *Common issues* table to support your own questioning.

You may want to use Slide P-5 of the projector resource, *Categorizing Equations*, to support the discussion.

Here are some questions you could ask the class:

What did you find difficult about this task? Which card did you find most difficult to place? Why?

Does anyone have two equation cards in the same cell?

Ben, where did you place this card? How did you decide?

Does anyone disagree with Ben? Did anyone use a different method from Ben's? Please explain your method. Which method do you prefer? Why?

You may also choose to focus on what students have learnt about the key features of the equation of a circle.

Do the equations you've placed in this row/column have anything in common?

What does this equation tell you about its graph?

Show me the equation of a circle with a radius 10 and a center at (-2, 1). How can you tell that this is the equation?

Suppose the equation of a circle is $(x - 3)^2 + (y + m)^2 = 12$*, where* m *is an integer. What is the radius of the circle? What are the coordinates of the center of this circle? Are there any other possible points for the center?*

Follow-up lesson: reviewing the assessment task (15 minutes)

Return the original assessment *Going Round in Circles* to the students and give them a copy of *Going Round in Circles (revisited)*. If you have not added questions to individual pieces of work or printed a list of questions, then write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

Look at your original responses and the questions (on the board/written on your script.) Use what you have learned to answer the questions

Now try to answer the questions on the new task sheet, Going Round in Circles (revisited).

Some teachers give this as a homework task.

SOLUTIONS

Assessment task: *Going Round in Circles*

1. a. The center of the circle is (0,0).

   b. Equation of the circle is $x^2 + y^2 = 36$.

   c. $m = \pm\sqrt{32}$ (or $\pm 4\sqrt{2}$). This may be seen using the Pythagorean theorem or by substituting into the equation of the circle.

2. Equation of the circle: $(x + 5)^2 + (y - 1)^2 = 14$.

Collaborative activity: *Making Posters*

	Center at (2,1)	Center at (2, –1)	Center at (0, –1)	Center (–2,1)
Radius of $\sqrt{5}$	**4.** $(y-1)^2 + (x-2)^2 = 5$	**11.** $(x-2)^2 + (y+1)^2 + 4 = 9$	$x^2 + (y+1)^2 = 5$	$(x+2)^2 + (y-1)^2 = 5$
Radius of $\sqrt{10}$	**7.** $(x-2)^2 + (y-1)^2 + 15 = 25$	**10.** $(x-2)^2 + (y+1)^2 = 10$	**9.** $(y+1)^2 + x^2 = 10$	**5.** $(x+2)^2 + (y-1)^2 = 10$
Radius of 5	**1.** $(x-2)^2 + (y-1)^2 = 25$	**12.** $(y+1)^2 + (x-2)^2 = 25$	**3.** $x^2 + (y+1)^2 = 25$	$(x+2)^2 + (y-1)^2 = 25$
Radius of 10	$(x-2)^2 + (y-1)^2 = 100$	**8.** $(x-2)^2 + (1+y)^2 = 100$	**6.** $x^2 + (y+1)^2 = 100$	**2.** $(x+2)^2 + (y-1)^2 - 100 = 0$

Assessment task: *Going Round in Circles (revisited)*

1. a. The center of the circle is (0,0).

   b. Equation of the circle is $x^2 + y^2 = 16$.

   c. $m = \pm\sqrt{12}$ (or $2\sqrt{3}$) Coordinates of points on the graph: $\left(\sqrt{12}, 2\right)$ or $\left(-\sqrt{12}, 2\right)$.

2. Equation of the circle: $(x - 4)^2 + (y + 2)^2 = 11$.

Going Round in Circles

You may want to use the space to the right of the questions to sketch graphs.

1. The end points of the diameter of a circle are (6, 0) and (−6, 0).

   a. What are the coordinates of the center of the circle?

   b. What is the equation of *this* circle?

   Fully explain your answer.

   c. A point on *this* circle has coordinates (2, *m*). Write possible values for *m*.

   Fully explain your answer.

2. The center of another circle is (−5, 1). Its radius is $\sqrt{14}$

   What is the equation of this circle?

   Fully explain your answer.

Card Set: Equations

1. $(x-2)^2+(y-1)^2=25$	2. $(x+2)^2+(y-1)^2-100=0$
3. $x^2+(y+1)^2=25$	4. $(y-1)^2+(x-2)^2=5$
5. $(x+2)^2+(y-1)^2=10$	6. $x^2+(y+1)^2=100$
7. $(x-2)^2+(y-1)^2+15=25$	8. $(x-2)^2+(1+y)^2=100$
9. $(y+1)^2+x^2=10$	10. $(x-2)^2+(y+1)^2=10$
11. $(x-2)^2+(y+1)^2+4=9$	12. $(y+1)^2+(x-2)^2=25$

Categorizing Equations

	Center at (2,1)	Center at (2,−1)	Center at (0,−1)	Center (__, __)
Radius of $\sqrt{5}$				
Radius of $\sqrt{10}$				
Radius of 5				
Radius of 10				

Going Round in Circles (revisited)

You may want to use the space to the right of the questions to sketch graphs.

1. The end points of the diameter of a circle are (−4, 0) and (4, 0).

   a. What are the coordinates of the center of the circle?

   b. What is the equation of *this* circle?

      Fully explain your answer.

   c. A point on *this* circle has coordinates (*m*, 2). Write possible values for *m*.

      Fully explain your answer.

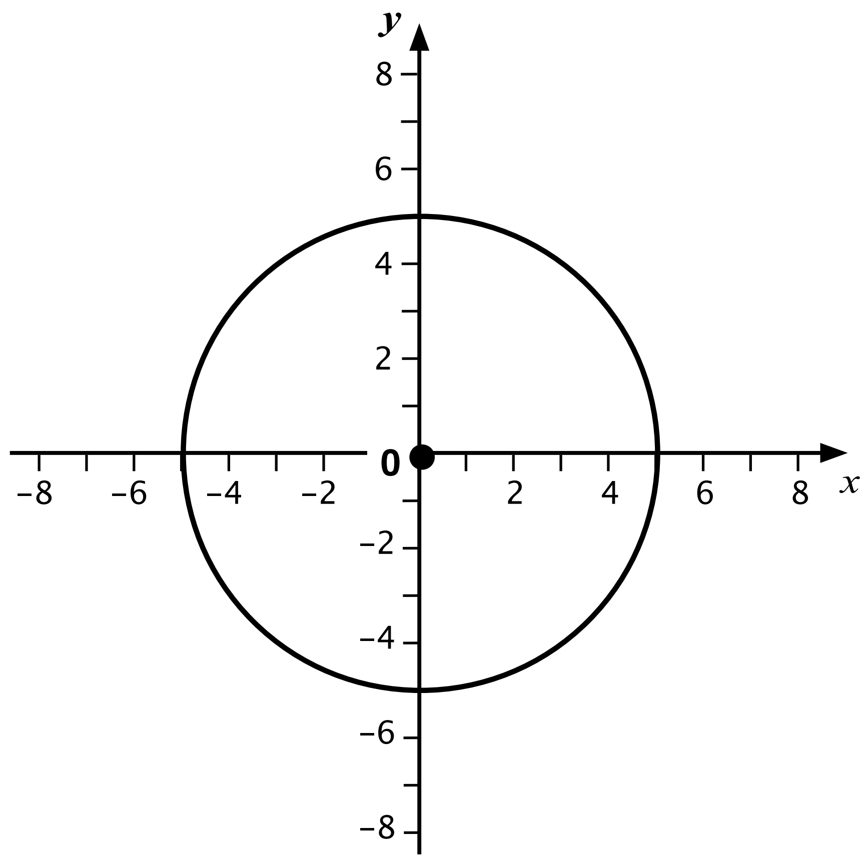
2. The center of another circle is (4,−2). Its radius is $\sqrt{11}$.

   What is the equation of this circle?

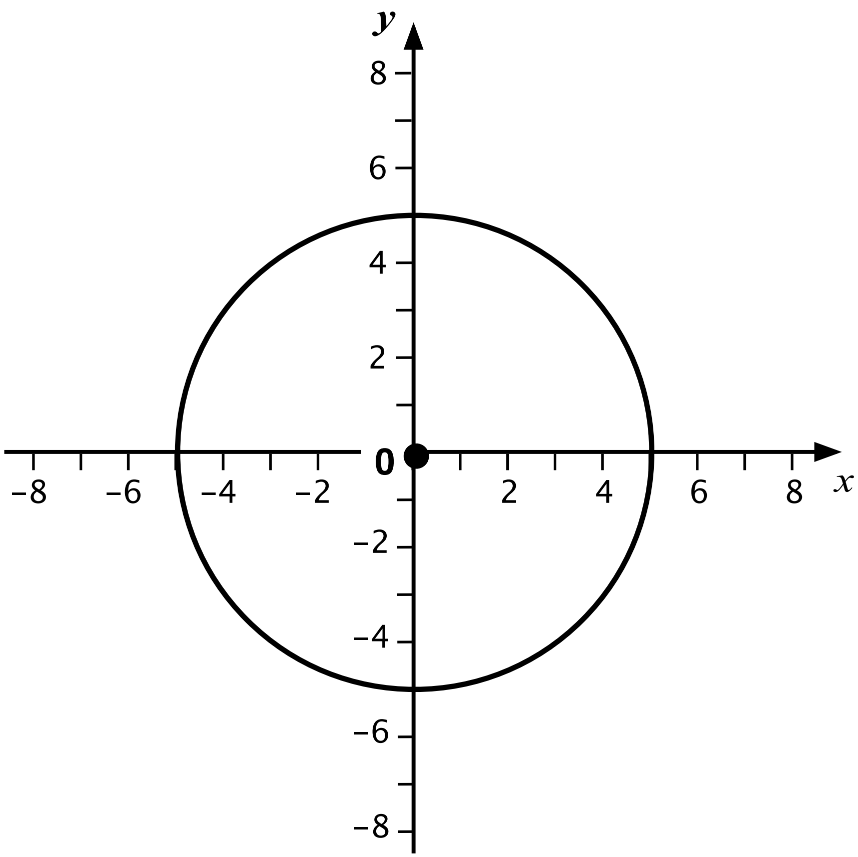
   Fully explain your answer.

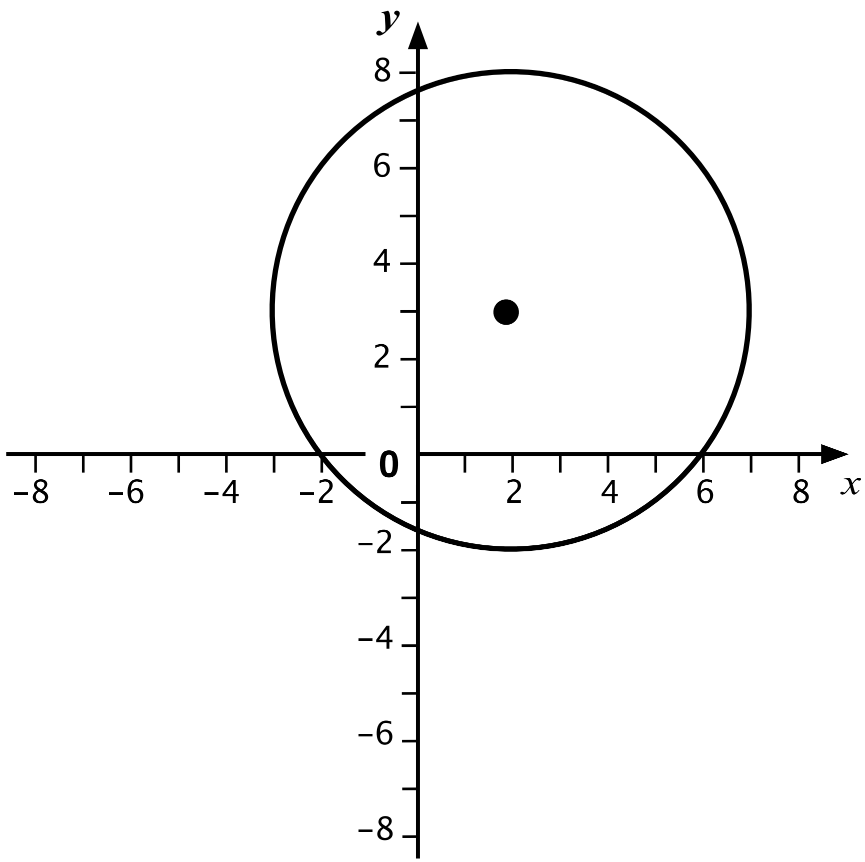
A circle with a center at (0,0) and a radius of 5

A point on the circle has co-ordinates $(3,y)$.
Write a value for y.



Two circles with the same radius but different centers





Categorizing Equations

1. Take turns to place an equation card in one of the categories in the table.
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3. If you don't agree or understand, ask your partner to explain their reasoning.
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5. Some of your cards are to go in one of the cells in the final column.
   - You will need to figure out the coordinates for the center of the circle for all equations placed in this column.
6. Make up your own equation for any empty cells.

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    – Go to another group's desk and check to see which cells are different from your own.

    – If there are differences, ask for an explanation. If you still don't agree, explain your own thinking.

    – When you return to your own desk you need to consider as a group whether to make any changes to your own table.

Categorizing Equations

	Center at (2,1)	Center at (2,−1)	Center at (0,−1)	Center (__, __)
Radius of $\sqrt{5}$				
Radius of $\sqrt{10}$				
Radius of 5				
Radius of 10				

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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We are grateful to the many teachers and students, in the UK and the US,
who took part in the classroom trials that played a critical role in developing these materials

The classroom observation teams in the US were led by
David Foster, Mary Bouck, and **Diane Schaefer**

This project was conceived and directed for
The Mathematics Assessment Resource Service (MARS) by
Alan Schoenfeld at the University of California, Berkeley, and
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Thanks also to Mat Crosier, Anne Floyde, Michael Galan, Judith Mills, Nick Orchard, and Alvaro Villanueva who contributed to the design and production of these materials

This development would not have been possible without the support of
Bill & Melinda Gates Foundation

We are particularly grateful to
Carina Wong, Melissa Chabran, and **Jamie McKee**

The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org