CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Representing Conditional Probabilities 1

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Representing Conditional Probabilities 1

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Understand conditional probability.
- Represent events as a subset of a sample space using tables and tree diagrams.
- Communicate their reasoning clearly.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

S-CP: Understand independence and conditional probability and use them to interpret data.

This lesson also relates to **all** the *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 2, 3, and 7:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

This lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task that is designed to reveal their current understanding and difficulties. You then review their work and create questions for students to answer in order to improve their solutions.
- At the start of the lesson, students work alone answering your questions about the same problem. Students are then grouped and engage in a collaborative discussion of the same task.
- In the same small groups, students are given sample solutions to analyze and evaluate.
- Finally, in a whole-class discussion, students explain and compare the alternative solution strategies they have seen and used.
- In a follow-up lesson students review what they have learnt.

MATERIALS REQUIRED

- Each student will need a copy of the *Lucky Dip* assessment task, the *How Did You Work?* questionnaire, a mini-whiteboard, pen, and eraser.
- Each small group of students will need a large sheet of paper for making a poster, a felt-tipped pen, and enlarged copies of the *Sample Responses to Discuss*.
- You will need a bag and some black and white balls (or some substitute) for a class demonstration.
- There is a projector resource to support whole-class discussion.

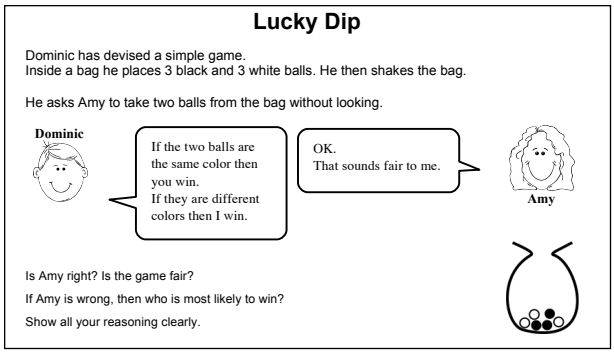
TIME NEEDED

20 minutes before the lesson, an 80-minute lesson (or two shorter lessons), and 10 minutes in a follow-up lesson. Exact timings will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Lucky Dip* (20 minutes)

Have students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the next lesson.



Give each student a copy of *Lucky Dip*. Make sure the class understands the rules of the game by demonstrating it using a bag and some black and white balls.

Read through the questions and try to answer them as carefully as you can.

It is important that students, as far as possible, are allowed to answer the questions without your assistance.

Students should not worry too much if they cannot understand or do everything, because in the next lesson they will engage in a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to answer questions such as these confidently. This is their goal.

Students who sit together often produce similar answers and then when they come to compare their work, they have little to discuss. For this reason, we suggest that when students do the task individually, you ask them to move to different seats. Then at the beginning of the formative assessment lesson, allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit. We suggest you make a list of questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these questions on the board when you return the work at the beginning of the lesson.

Common issues	Suggested questions and prompts
Has difficulty getting started	• Play the game twenty times, using pieces of marked paper instead of balls. Do you think the game is fair? Explain your answer.
Assumes that the game is fair For example: The student assumes that there are only two outcomes (the balls are the same color or the balls are different colors), so the probabilities are equal.	• Suppose you labeled each ball with a different letter. What are the different combinations you can take out of the bag?
Uses unsuitable representations For example: The student does not use a tree diagram or a sample space diagram.	• Can you think of a suitable diagram that will show all the possible outcomes? • Can you use a sample space diagram?
Confusion concerning the nature of the 'event' For example: The student is concerned over whether it makes a difference to consider taking both balls out at once or taking them out one at a time.	• Does it make a difference whether Amy picks the balls one at a time, rather than at the same time? Explain your answer. • How can you show the different possible outcomes using a diagram?
Dependent probabilities are not recognized For example: The student appears to assume that each ball is returned to the bag after it is selected. Or: The probability that the second ball is black is assumed to be independent of the choice of the first ball. So P(both balls black) is assumed to be 0.5×0.5.	• Imagine picking a black ball from the bag. What is the probability of picking a black ball? Now you are holding the black ball, what is the probability of picking another black ball?
The same ball is selected twice in the table of possible outcomes For example: The student assumes that there are $3 \times 3 = 9$ ways of obtaining two black balls.	• Is it possible to select the same ball twice?
Work is incomplete or unclear For example: The student does not fully label the tree diagram or the sample space diagram.	• Would someone unfamiliar with this type of task understand all your work?
All questions are answered correctly The student needs an extension task.	• How many black balls and how many white balls could you put in the bag to make the game fair? Explain your answer.

SUGGESTED LESSON OUTLINE

Individual work (10 minutes)

Begin the lesson by briefly reintroducing the problem.

Recall what we were looking at in a previous lesson. What was the task about?

Return the assessment task to the students. Give each student a mini-whiteboard, pen, and eraser.

If you did not add questions to individual pieces of work, write your list of questions on the board or give out your printed list. Students are to select questions appropriate to their own work and spend a few minutes answering them.

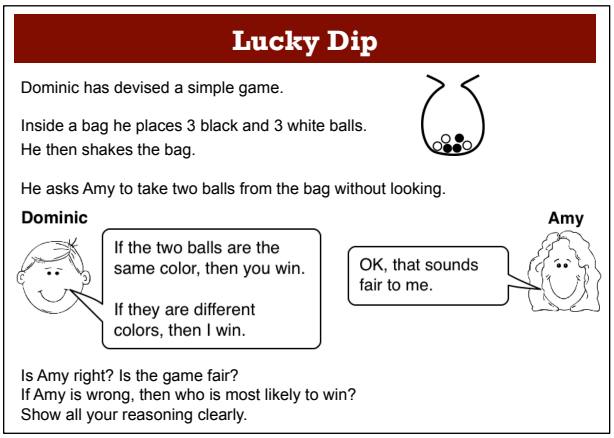
Today we are going to work together to try to improve your initial attempts at this task.

I have looked at your work and I have some questions I would like you to think about.

On your own, carefully read through the questions I have written. Use the questions to help you to think about ways of improving your own work.

Use your mini-whiteboards to make a note of anything you think will help to improve your work.

Slide P-1 of the projector resource outlines the rules of the game. To remind students of the rules, you could demonstrate the game using a bag and some balls.



Collaborative small-group work (20 minutes)

Organize the class into small groups of two or three students. Give each group a large piece of paper and a felt-tipped pen.

Deciding on a Strategy

Ask students to share their ideas about the task and plan a joint solution.

I want you to share your work with your group.

Take turns to explain how you did the task and how you now think it could be improved.

Listen carefully to any explanation. Ask questions if you don't understand or agree with the method. (You may want to use some of the questions I have written on the board.)

I want you to plan a joint approach that is better than your separate solutions.

Once students have evaluated the relative merits of each approach, ask them to write their strategy on the second side of the poster.

Slide P-2 of the projector resource, *Planning a Joint Solution*, summarizes these instructions:

Planning a Joint Solution

1. Take turns to explain how you did the task and how you now think it could be improved.

2. Listen carefully to explanations.
   - Ask questions if you don't understand.
   - Discuss with your partner(s):
     - What you like/dislike about your partner's math.
     - Any assumptions your partner has made.
     - How their work could be improved.

3. Once everyone in the group has explained their solution, plan a joint approach that is better than each of the separate solutions.
   - On the second side of your poster or paper write a couple of sentences outlining your plan.

Implementing the Strategy

Students are now to write their joint solution on the front side of the poster.

While students work in small groups you have two tasks: to note different student approaches to the task and to support student problem solving.

Note different student approaches to the task

For example, do students identify all the different possible events clearly? Are students using diagrams to support their answers? In particular, note any common mistakes. You can then use this information to focus a whole-class discussion at the end of the lesson.

Support student problem solving

Try not to make suggestions that move students towards a particular approach to the task. Instead, ask questions that help students to clarify their thinking. In particular focus on the strategies rather than the solution. Encourage students to justify their ideas.

Look for any groups of students who agree amongst themselves on an incorrect answer or justification. You could ask these students to work with another group, to compare solutions and prompt revision.

You may want to use the questions in the *Common issues* table to support your own questioning. If the whole-class is struggling on the same issue, you could write one or two relevant questions on the board and hold a brief whole-class discussion. You could also give any struggling students one of the *Sample Responses to Discuss*.

Whole-class discussion (15 minutes)

You may want to hold a brief whole-class discussion. Have students solved the problem using a variety of methods? Or have you noticed some interesting ways of working or some incorrect methods, if so, you may want to focus the discussion on these. Equally, if you have noticed different groups using similar strategies but making different assumptions you may want to compare solutions.

Extending the lesson over two days

If you are taking two days to complete the lesson unit then you may want to end the first lesson here. At the start of the second day, allow students time to familiarize themselves with their joint solution before moving on to the collaborative analysis of sample responses.

Collaborative analysis of *Sample Responses to Discuss* (20 minutes)

After students have had enough time to attempt the problem, give each group copies of the three *Sample Responses to Discuss* and ask for written comments. This task gives students the opportunity to evaluate a variety of approaches to the task.

You are now going to look at three solutions to the task.

Imagine you are the teacher. Write down your comments on each piece of work.

Try to explain what the student has done.

What mistakes have been made?

What isn't clear about the work?

Slide P-3 of the projector resource describes how students should work together:

Evaluating Sample Responses to Discuss

1. Imagine you are the teacher and have to assess the student work.
2. Take turns to work through a students' solution.
   - Write your answers on your mini-whiteboards.
3. Explain your answer to the rest of the group.
4. Listen carefully to explanations.
   - Ask questions if you don't understand.
5. Once everyone is satisfied with the explanations, write the answers below the students' solution.
   - Make sure the student who writes the answers is not the student who explained them.

During the paired work, support the students as in the first collaborative activity. Note similarities and differences between the approaches seen in the sample responses and those students took in the small-group work. Also, check to see which methods students have difficulties in understanding. This information can help you focus the next activity, a whole-class discussion.

Whole-class discussion: comparing different approaches (15 minutes)

Hold a whole-class discussion to consider the different approaches used in the sample work. Focus the discussion on parts of the task students found difficult. Ask the students to compare the different solution methods:

Which approach did you like best? Why?

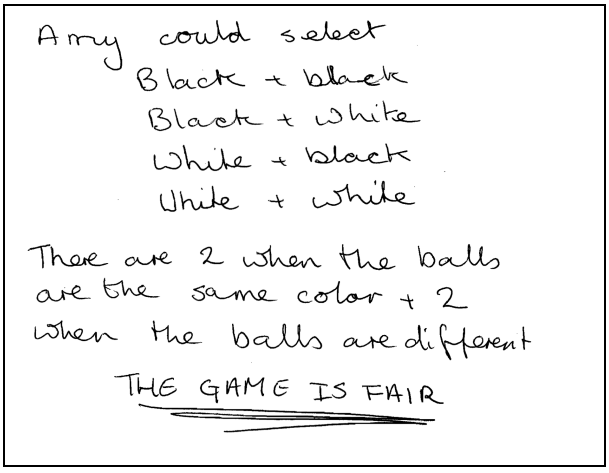
Which approach did you find most difficult to understand? Why?

To support the discussion, you may want to use Slides P-4 to P-6 of the projector resource to display the sample work.

Anna's work appears intuitively correct.

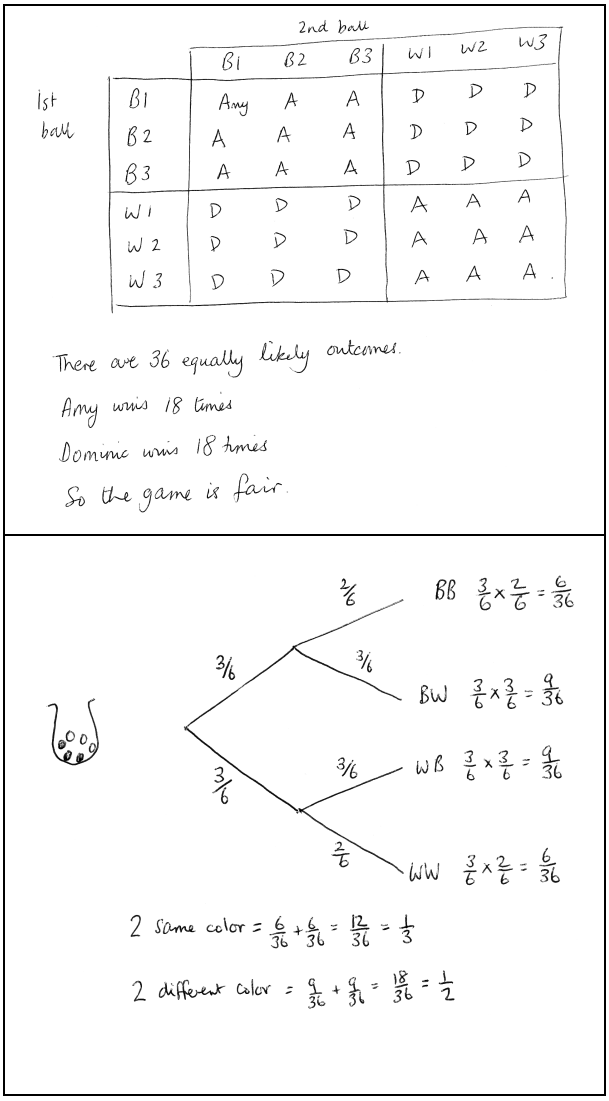
She assumes that there are only two outcomes (that the two balls are the same color or that they are different colors), so that the probabilities are equal.

Anna does not take into account the changes in probabilities once a ball is removed from the bag and not replaced.



Ella draws a sample space in the form of an organized table.

Ella clearly presents her work, however she makes the mistake of including the diagonals. This means the same ball is selected twice. This is not possible, as the balls are not replaced.



Jordan uses a tree diagram to show the possible outcomes when taking two balls from the bag.

Jordan's work is difficult to follow. He does not label the branches of the tree.

Jordan does not take into account that the first ball is not replaced. When selecting the second ball there are only 5 balls in the bag, so these probability fractions should all have a denominator of 5.

Where does the denominator of 6 come from?

What is the sum of all final probabilities?
What does this tell you about Jordan's work?
[He has made a mistake.]

Follow-up lesson: review solutions to *Lucky Dip* (10 minutes)

Give out the *How Did You Work?* sheet and ask students to complete this questionnaire.

The questionnaire should help students review their progress.

If you have time, you may also want to ask your students to read through their original solution and using what they have learned, attempt the task again. In this case, give each student a fresh blank copy of the *Lucky Dip* assessment.

Some teachers give this as a homework task.

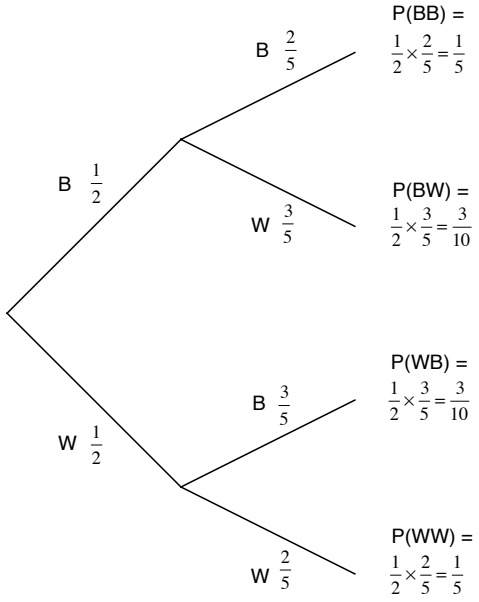
SOLUTIONS

Assessment task: *Lucky Dip*

Amy is wrong: the game is not fair. In the sample space diagram below, the black balls are labeled B_1, B_2, B_3 and the white balls are labeled W_1, W_2, W_3. Each cell shows one possible, equally likely outcome. The diagonal doesn't show possible outcomes because the same ball cannot be taken out twice.

Amy wins wherever there is a ✓. Dominic wins wherever there is a x. This shows that the probability of Amy winning is $\frac{12}{30} = \frac{2}{5}$ and the probability of Dominic winning is $\frac{18}{30} = \frac{3}{5}$. An alternative representation is the tree diagram.

		First selection					
		B_1	B_2	B_3	W_1	W_2	W_3
Second selection	B_1	-	✓	✓	x	x	x
	B_2	✓	-	✓	x	x	x
	B_3	✓	✓	-	x	x	x
	W_1	x	x	x	-	✓	✓
	W_2	x	x	x	✓	-	✓
	W_3	x	x	x	✓	✓	-



Some students think of the event being modeled as **picking two balls simultaneously**. In that case, the sample space diagram (with labels first selection, second selection) and the probability tree (which again shows a sequence of events) may seem less appropriate. The student may therefore decide to not distinguish between B1B2 and B2B1. The resulting sample space diagram will be just the upper (or lower) half of the sample space diagram shown above. The resulting probabilities however will remain unaffected.

Lucky Dip

Dominic has devised a simple game.
Inside a bag he places 3 black and 3 white balls. He then shakes the bag.

He asks Amy to take two balls from the bag without looking.

Dominic

If the two balls are the same color then you win.
If they are different colors then I win.

OK.
That sounds fair to me.

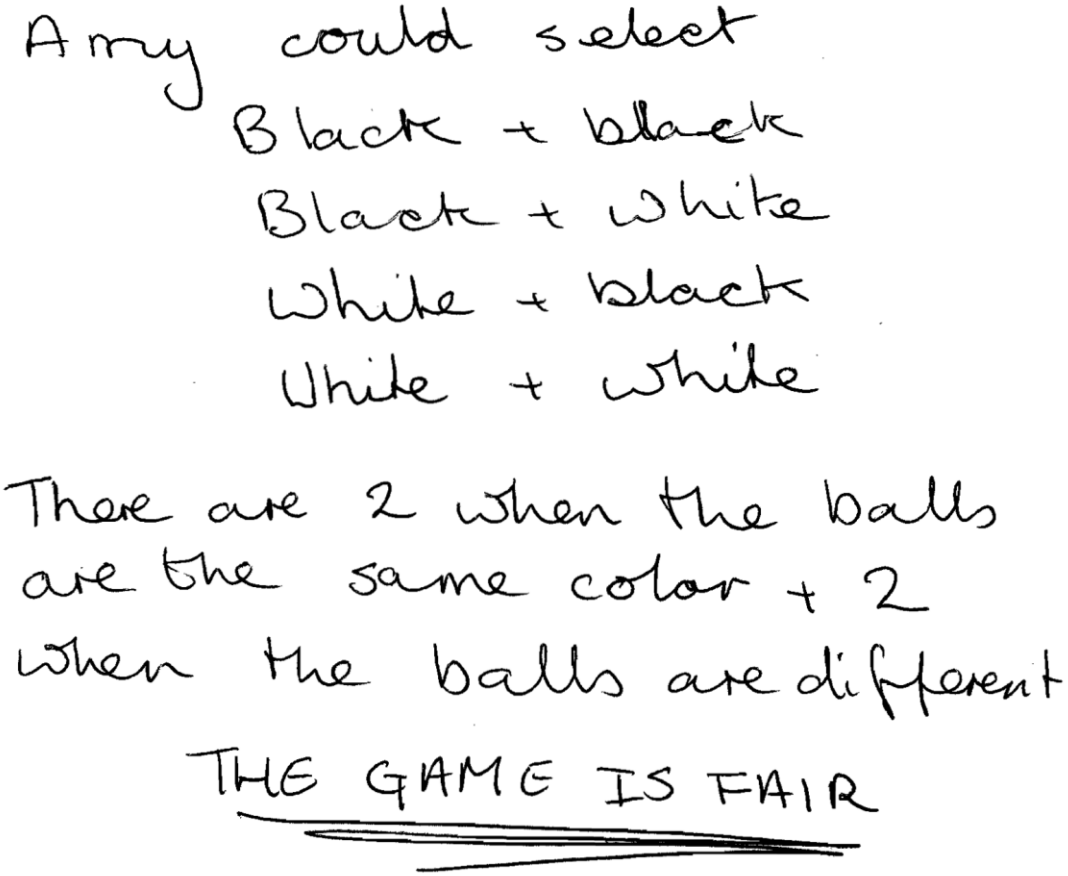
Amy

Is Amy right? Is the game fair?

If Amy is wrong, then who is most likely to win?

Show all your reasoning clearly.

Sample Responses to Discuss: Anna

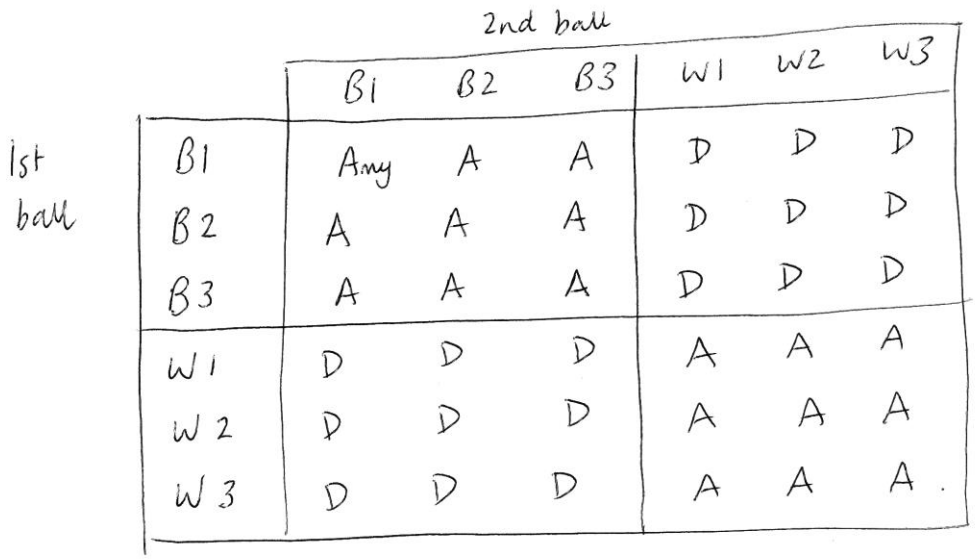


Explain what the student has done.

What isn't clear about her work?

What mistakes has she made?

Sample Responses to Discuss: Ella



1st ball \ 2nd ball	B1	B2	B3	W1	W2	W3
B1	Amy	A	A	D	D	D
B2	A	A	A	D	D	D
B3	A	A	A	D	D	D
W1	D	D	D	A	A	A
W2	D	D	D	A	A	A
W3	D	D	D	A	A	A

There are 36 equally likely outcomes.

Amy wins 18 times

Dominic wins 18 times

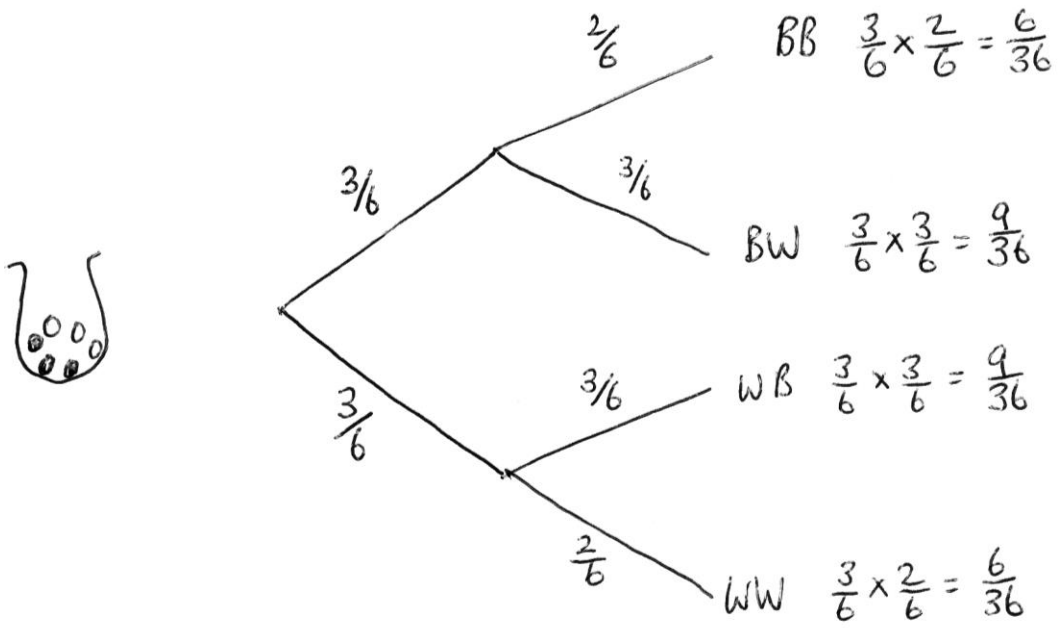
So the game is fair.

Explain what the student has done.

What isn't clear about her work?

What mistakes has she made?

Sample Responses to Discuss: Jordan



2 same color = $\frac{6}{36} + \frac{6}{36} = \frac{12}{36} = \frac{1}{3}$

2 different color = $\frac{9}{36} + \frac{9}{36} = \frac{18}{36} = \frac{1}{2}$

Explain what the student has done.

What isn't clear about his work?

What mistakes has he made?

How Did You Work?

1. Compare the sample responses and your group response. What are the advantages and disadvantages of each approach?

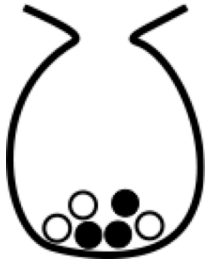
	Advantages	Disadvantages
Amy		
Ella		
Jordan		
Our group work		

2. Now that you have seen Amy's, Ella's and Jordan's work, what would you do if you started the task again?

3. What do you think are the difficulties someone new to the task will face?

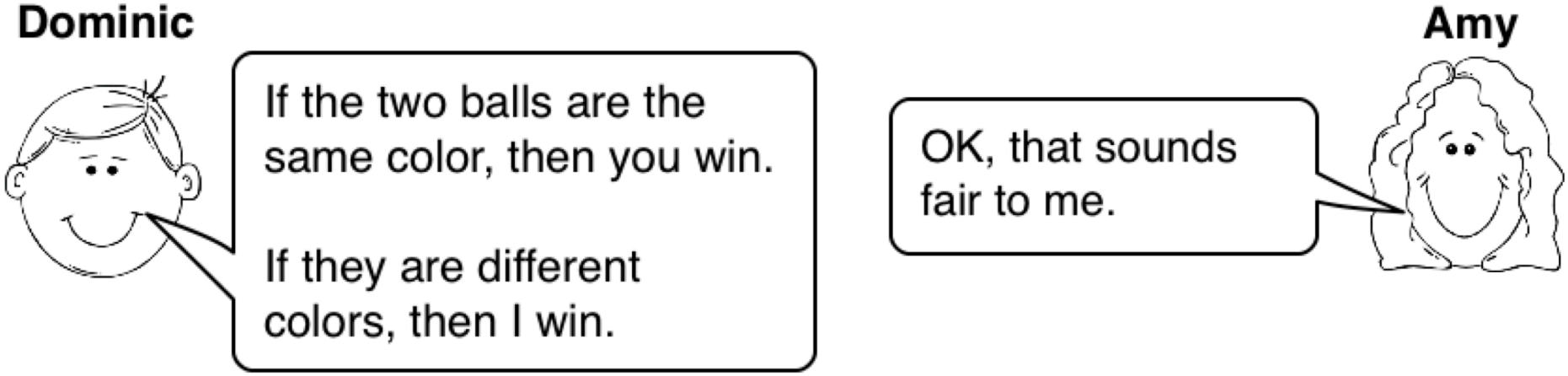
Lucky Dip

Dominic has devised a simple game.



Inside a bag he places 3 black and 3 white balls.
He then shakes the bag.

He asks Amy to take two balls from the bag without looking.



Is Amy right? Is the game fair?
If Amy is wrong, then who is most likely to win?
Show all your reasoning clearly.

Planning a Joint Solution

1. Take turns to explain how you did the task and how you now think it could be improved.

2. Listen carefully to explanations.
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Evaluating Sample Responses to Discuss

1. Imagine you are the teacher and have to assess the student work.

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5. Once everyone is satisfied with the explanations, write the answers below the students' solution.
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Sample Responses to Discuss: Anna

Amy could select

Black + black

Black + White

White + black

White + White

There are 2 when the balls are the same color + 2 when the balls are different

THE GAME IS FAIR

Sample Responses to Discuss: Ella

1st ball \ 2nd ball	B1	B2	B3	W1	W2	W3
B1	Amy	A	A	D	D	D
B2	A	A	A	D	D	D
B3	A	A	A	D	D	D
W1	D	D	D	A	A	A
W2	D	D	D	A	A	A
W3	D	D	D	A	A	A

There are 36 equally likely outcomes.

Amy wins 18 times

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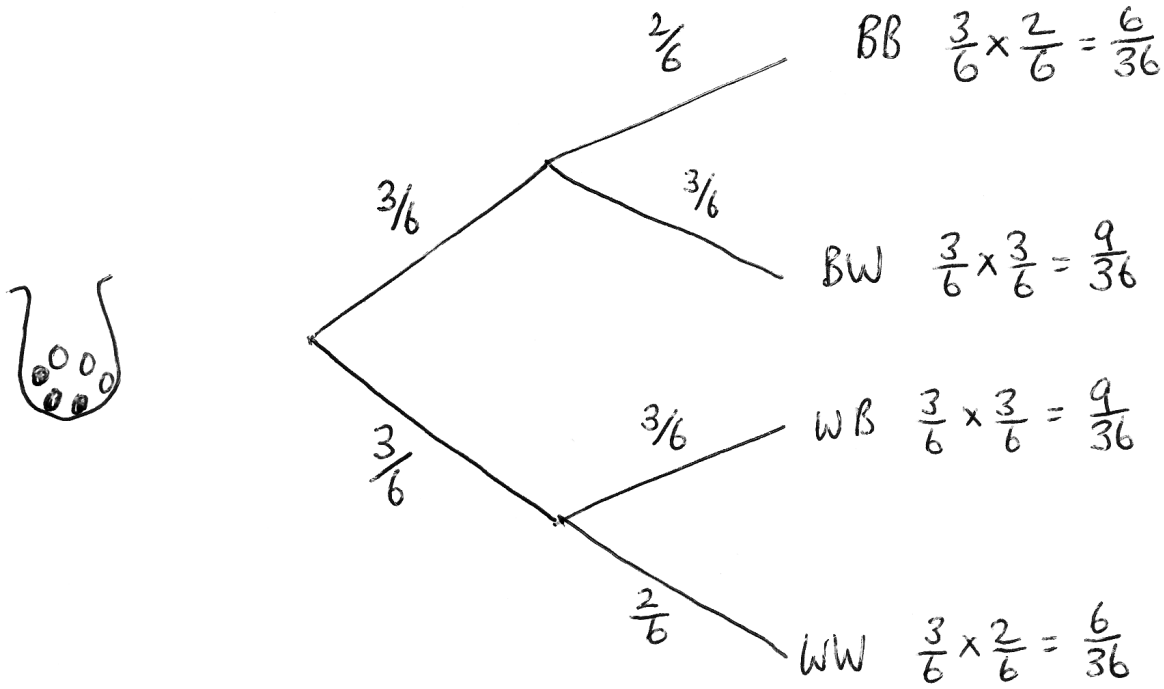
So the game is fair.

Sample Responses to Discuss: Jordan



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Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org